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ORIGINAL ARTICLE

# BOOTSTRAP PROCEDURES FOR DETECTING MULTIPLE PERSISTENCE SHIFTS IN HETEROSKEDASTIC TIME SERIES

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This article proposes new bootstrap procedures for detecting multiple persistence shifts in a time series driven by non-stationary volatility. The assumed volatility process can accommodate discrete breaks, smooth transition variation as well as trending volatility. We develop wild bootstrap sup-Wald tests of the null hypothesis that the process is either stationary [$I(0)$] or has a unit root [$I(1)$] throughout the sample. We also propose a sequential procedure to estimate the number of persistence breaks based on ordering the regime-specific bootstrap $p$-values. The asymptotic validity of the advocated procedures is established both under the null of stability and a variety of persistence change alternatives. A comparison with existing tests that assume homoskedasticity illustrates the finite sample improvements offered by our methods. An application to OECD inflation rates highlights the empirical relevance of the proposed approach and weakens the case for persistence change relative to existing procedures.

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## 1. INTRODUCTION

Distinguishing among different forms of non-stationarity has been a topic of long-standing interest in time series analysis; for example, distinguishing a deterministic trend from a unit root process ($I(1)$, a stochastic trend) and infrequent shifts in trend versus unit root (Perron, 1989). Also, tests for changes in the deterministic trend are sensitive to the nature of the stochastic component, stationary [$I(0)$] or $I(1)$ (Harvey *et al*., 2009; Perron and Yabu, 2009). An important related problem concerns inference about the conditional mean in the presence of non-stationarity in variance. Structural changes in variance have been extensively documented for macroeconomic and financial time series; for example, Sensier and van Dijk (2004), Perron and Yamamoto (2020). The non-robustness of unit root tests to non-stationary volatility was established by Cavaliere (2005) and Cavaliere and Taylor (2007, 2008a, 2009). A smaller literature addressed the problem of discriminating between changes in the conditional mean and non-stationary volatility. Hansen (2000) shows that standard structural change tests do not have the correct size with non-stationary variance. Pitarakis (2004), Perron and Yamamoto (2019) and Xu (2015) document the extent of size distortions and power losses for various tests. Perron *et al*. (2020) develop likelihood ratio tests of the joint hypothesis of changes in coefficients and error variance in a linear regression model.

Given the importance of allowing for non-stationary volatility, this article deals with the problem of testing for structural changes in the persistence of a time series in this context; that is, changes involving switches between unit root $I(1)$ and stationary $I(0)$ processes and changes that preserve the $I(0)$ properties across regimes. Most

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procedures available are based on a global homoskedasticity assumption; for example, Kim (2000), Busetti and Taylor (2004), Harvey *et al*. (2006) for a single break and Bai and Perron (1998, BP henceforth), Leybourne *et al*. (2007), Kejriwal *et al*. (2013, KPZ henceforth) for multiple breaks as well as Kejriwal (2019) for procedures to determine the number of breaks. Cavaliere and Taylor (2008b, CT henceforth) develop bootstrap tests robust to non-stationary volatility based on the ratio of partial sums of demeaned (or detrended) residuals. Their procedure assumes that the process is $I(0)$ under the null hypothesis of stability, a single break under the alternative with either a $I(1) - I(0)$ or $I(0) - I(1)$ shift but not an $I(0) - I(0)$ shift and a stable trend function.

We provide a comprehensive treatment of issues related to testing for changes in persistence with heteroskedastic errors. Our approach is general and allows: (i) an $I(1)$ or $I(0)$ null hypothesis; (ii) multiple changes with unknown number and timing; (iii) changes of the form $I(1)-I(0)$, $I(0)-I(1)$ and $I(0)-I(0)$, without prior knowledge of the specific form; (iv) disentangling persistence shifts from shifts in the trend function. The assumed volatility process is general and accommodates breaks, smooth transitions and trending volatility. We develop sup-Wald tests based on a wild bootstrap scheme that have accurate size and satisfactory power in finite samples. We also propose a sequential method to estimate the number of persistence breaks. Extensive simulation experiments are provided to assess the finite sample properties of the methods suggested, including comparisons with existing tests.

Our proposed methods can be applied to study a wide range of important empirical issues. We comment here on three potential applications, one explored in detail later. The first concerns the issue of inflation persistence that plays a key role in the formulation and evaluation of quantitative macroeconomic models (see, e.g., Korenok *et al*., 2010). The Lucas Critique suggests that the parameters of reduced form specifications depend implicitly on agents' expectations of the policy process and are unlikely to remain stable as policymakers change their behavior, if agents are forward looking. An empirical finding of high and stable persistence in such a context can potentially be interpreted either in terms of the presence of a strong backward looking component in the dynamics of inflation induced through, say indexation or rule-of-thumb behavior on the part of the price setters, or in terms of historical policy shifts being of relatively modest magnitude. Our approach enables a robust treatment of inflation dynamics that allows disentangling breaks in mean and persistence allowing for changing volatility and thereby provides reliable guidance on whether changes in persistence is a feature that a reasonable macroeconomic model should be able to replicate; see Section 8 for the analysis, references and further discussions.

A second application concerns climate change. The anthropogenic theory of climate change postulates that human activity increases emissions of radiatively active gases relative to natural sources and sinks. It can do so in a way that changes global biogeochemical cycles thereby increasing the persistence of radiative forcing and surface temperature. Using data over 1500–2011, Dergiades *et al*. (2016) find evidence supporting persistence change [$I(0)$ to $I(1)$] for both series based on the single break tests of Busetti and Taylor (2004) and Harvey *et al*. (2006). The long time span, however, covers the three Industrial Revolutions (the steam engine, electricity and mass production, and digital technology) suggesting that the single break assumption may be unduly restrictive. Furthermore, they assume homoskedasticity in contrast to evidence favoring heteroskedasticity in both series (see Cavaliere *et al*., 2018, for global $CO_2$ emissions and Chang *et al*., 2020, for temperature). Our approach offers methods to comprehensively evaluate this hypothesis owing to its ability to accommodate multiple persistence breaks, non-stationary volatility, and broken trends (Estrada and Perron, 2017).

A third possible application is to predictive regression. Predicting a low persistence $I(0)$ variable like stock returns using a highly persistent predictor, say the dividend–price ratio ($D/P$), involves a (nearly) unbalanced regression that can potentially explain its sporadic predictive power in practice. Lettau and Nieuwerburgh (2008) argue that $D/P$ is well represented by a regime-wise stationary process and use demeaned residuals obtained from the BP procedure to construct a new predictor that is shown to deliver superior predictive performance (see Verdickt *et al*., 2019, for a more recent related contribution). In contrast, Park (2010) argues, based on single break homoskedasticity-based persistence change tests (Harvey *et al*., 2006), that $D/P$ is better approximated by a process that switches between $I(1)$ and $I(0)$ regimes and consequently has strong predictive power in the $I(0)$ regime but not in the $I(1)$ regime due to the unbalanced regression problem. Given that neither of these studies allow for non-stationary volatility in $D/P$ or returns (Johannes *et al*., 2014), their estimation and inference results can be potentially misleading. The generality afforded by our approach can be fruitfully employed in this context




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to distinguish between the mean shift and persistence change alternatives for the $D/P$ process, demarcate the $I(1)$ and $I(0)$ regimes, and assess regime-wise predictability accordingly.

The article is organized as follows. Section 2 describes the models and the assumptions and Section 3 the testing procedures. The large sample effects of non-stationary volatility on these persistence change tests are studied in Section 4. The proposed bootstrap tests are presented in Section 5. Section 6 discusses extensions of the procedures to deal with deterministic trends as well as disentangling shifts in persistence from shifts in the trend function. Section 7 provides a summary of the Monte Carlo evidence, Section 8 presents an application to OECD inflation rates and Section 9 concludes. An online supplement contains all proofs, detailed simulation results and additional empirical results.

## 2. THE PERSISTENCE CHANGE MODEL

We start with a univariate time series $y_t$ generated by the following $AR(p)$ model:

$$
\begin{aligned}
y_t &= \mu_i + u_t; \quad u_t = u_{T_{i-1}^0} + h_t \ (t = T_{i-1}^0 + 1, \ldots, T_i^0;\ i = 1, \ldots, m+1) \\
h_t &= \alpha_i h_{t-1} + \sum_{j=1}^{p-1} \pi_{ij} \Delta h_{t-j} + e_t \ (h_{T_{i-1}^0} = \cdots = h_{T_{i-1}^0 - p+1} = 0)
\end{aligned}
\tag{1}
$$

with the convention $T_0^0 = 0$ and $T_{m+1}^0 = T$, where $T$ is the sample size. The process is therefore subject to $m$ breaks or $m+1$ regimes with break dates $(T_1^0, \ldots, T_m^0)$. Both the break dates and the number of breaks are assumed unknown. The autoregressive order $p$ is assumed to be finite. This data generating process (DGP) was considered by Leybourne *et al*. (2007) and Kejriwal (2019) and is designed to ensure that adjacent $I(1)$ and $I(0)$ regimes join up at the breakpoints thereby avoiding the problem of spurious jumps to zero in $u_t$. While we assume a common lag length $p$ across regimes, regime-specific lag lengths can be accommodated interpreting $p$ as the maximum lag length across the $(m+1)$ regimes. Our analysis is based on the following assumptions:

**Assumption 1.** (1) $T_i^0 = \lfloor T\lambda_i^0 \rfloor$, where $0 < \lambda_1^0 < \cdots < \lambda_m^0 < 1$ and $\lfloor . \rfloor$ denotes the integer part of its argument; (2) all roots of the polynomial $\pi_i(L) = 1 - \pi_{i1}L - \pi_{i2}L^2 - \cdots - \pi_{i,p-1}L^{p-1}$ lie outside the unit circle; (3) $e_t = \sigma_t \varepsilon_t$, where $\{\varepsilon_t\}$ is an i.i.d. sequence with zero mean and unit variance and $\{\sigma_t\}$ is a strictly positive non-stochastic sequence, also $\sup_t E(\varepsilon_t^{4+\beta}) < \infty$ for some $\beta > 0$; 4) For some strictly positive deterministic sequence $\{a_T\}$, $\{\sigma_t\}$ satisfies $a_T^{-1}\sigma_{\lfloor Ts \rfloor} = g(s)$, $s \in [0, 1]$, where $g(.)$ is a strictly positive, non-stochastic function with a finite number of discontinuities satisfying a uniform first-order Lipschitz condition except at the points of discontinuity.

Assumption A1 is standard and dictates the asymptotic framework adopted so that each segment increases proportionately with $T$. A2 specifies at most one unit root in each regime and precludes explosive regimes. A3 specifies that the stochastic process for $\{e_t\}$ is determined by the time-varying volatilities $\{\sigma_t\}$ (e.g., Xu, 2008). In contrast to CT who make a mixing-type assumption on the errors thereby allowing for moving average processes and conditional heteroskedasticity, our analysis is based on the stronger assumption of a finite order autoregressive process with i.i.d. innovations. We do, however, demonstrate the robustness of our procedures to moving average errors through simulations (see Supplement B). A4 is the key assumption, which allows $\{y_t\}$ to be generated by a wide class of non-stationary heteroskedastic errors; for example, single or multiple volatility breaks, linearly trending volatility, piece-wise linear trends in variance and smooth transition shifts satisfy A4 with $a_T = 1$ for a particular choice of $g(.)$. Models with explosive deterministic volatility are allowed specifying $a_T$ appropriately; see Cavaliere and Taylor (2008a, 2009). The function $g(.)$ is assumed to be non-stochastic to enable simplification of the theoretical analysis. Hence, A4 rules out non-stationary autoregressive stochastic volatility (SV) models (Hansen, 1995), SV models with jumps (Georgiev, 2008, Perron and Qu, 2010), 'non-stationary nonlinear heteroskedastic' models with stochastically trending volatility, and near-integrated GARCH models. This assumption can be weakened to allow sequences $\{\sigma_t\}$ and $\{\varepsilon_t\}$ that are stochastically independent and interpreting the results as holding conditional on a given realization of $g(.)$.

  
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To accommodate $I(0)$ preserving persistence changes as in the framework of Bai and Perron (1998), we also consider the following data generating process for $y_t$:

$$y_t = \mu_i + u_t; \quad u_t = \alpha_i u_{t-1} + \sum_{j=1}^{p-1} \pi_{ij} \Delta u_{t-j} + e_t \tag{2}$$

for $t = T_{i-1}^0 + 1, \ldots, T_i^0$; $i = 1, \ldots, m+1$, with $u_0 = \cdots = u_{-p+1} = 0$. The conditions in Assumption A are assumed to hold for (2) as well.

## 3. TESTING PROCEDURES

This section describes the testing procedures which form the basis of our proposed bootstrap tests. These procedures are not new and were developed in KPZ and BP. We first consider the KPZ tests that specify the null hypothesis of an $I(1)$ process throughout the sample (no change), $H_0^{(1)}$: $\alpha_i = 1$, versus the alternative that the process switches between being $I(1)$ and $I(0)$. The following two models are considered depending on whether the initial regime contains a unit root or not: Model 1a: $\alpha_i = 1$ in odd regimes and $|\alpha_i| < 1$ in even regimes; Model 1b: $\alpha_i = 1$ in even regimes and $|\alpha_i| < 1$ in odd regimes. We next review the test statistics designed to detect a specified number of breaks and outline the procedures when the number of breaks is not specified.

### 3.1. Tests for a Specified Number of Breaks

To test the null hypothesis $H_0^{(1)}$: $\alpha_i = 1$ for all $i$ in (1), the regression used is

$$\Delta y_t = c_i + (\alpha_i - 1) y_{t-1} + \sum_{j=1}^{p-1} \pi_j \Delta y_{t-j} + e_t^* \tag{3}$$

with $c_i = (1 - \alpha_i)[u_{T_{i-1}^0} + \mu_i]$ and $e_t^*$ the residuals. Under $H_0^{(1)}$, $c_i = 0$ for all $i$, which is imposed. The true lag order $p$ is assumed known but can be estimated using standard information criteria such as the Akaike information criterion (AIC) or BIC. The coefficients of the lagged differences in (3) are not allowed to change since, as argued in KPZ, the goal is to direct power against changes in the persistence parameter $\alpha_i$. A joint test on all parameters would not be particularly informative in this context given the difficulty in interpreting a rejection. As shown in KPZ, the test does not have much power against pure changes in short-run dynamics but is powerful when there is a change in both persistence and these dynamics. In fact, the simulation experiments conducted in Section 7 for assessing power with serially correlated errors consider DGPs that involve changes in both persistence and short-run dynamics in their autoregressive representation.

Consider first the Wald test that applies when the alternative involves a fixed value $m = k$ of changes. Denote a candidate vector of break fractions by $\lambda = (\lambda_1, \ldots, \lambda_k)$ and the alternative hypotheses corresponding to Models 1a and 1b as $H_{a,k}^{(1)}$ and $H_{b,k}^{(1)}$ respectively. The corresponding tests are for $d = a, b$:

$$F_{1d}(\lambda, k) = (T - k - 1 - ij)(SSR_0^{(1)} - SSR_{1d,k}^{(1)}) / [(k + 1 + ij) SSR_{1d,k}^{(1)}] \tag{4}$$

where $j = 1$ if $k$ is even and $j = 0$ if $k$ is odd, $i = -1$ if $d = a$ and $i = 1$ if $d = b$. Here, $SSR_0^{(1)}$ is the sum of squared residuals (SSR) under $H_0^{(1)}$ while $SSR_{1a,k}^{(1)}$ and $SSR_{1b,k}^{(1)}$ denote respectively, the SSR from estimating (3) under the restrictions imposed by Models 1a and 1b. For some small positive number $\epsilon$, we define the set $\Lambda_\epsilon^k = \{\lambda : |\lambda_{i+1} - \lambda_i| \geq \epsilon, \lambda_1 \geq \epsilon, \lambda_k \leq 1 - \epsilon\}$. The sup-Wald tests are then $F_{1a}(k) = \sup_{\lambda \in \Lambda_\epsilon^k} F_{1a}(\lambda, k)$ and $F_{1b}(k) = \sup_{\lambda \in \Lambda_\epsilon^k} F_{1b}(\lambda, k)$. When the initial regime is unknown, the relevant test statistic is $W_1(k) = \max[F_{1a}(k), F_{1b}(k)]$. The stable $I(0)$ null can be tested using the BP procedure. This amounts to testing $H_0^{(0)}$: $c_i = c$, $\alpha_i = \alpha$, for all $i$ with $|\alpha| < 1$ in (3). The relevant alternative hypothesis is $H_{1,k}^{(0)}$: $\alpha_1 \neq \alpha_2 \neq \cdots \neq \alpha_{k+1}$, $|\alpha_i| < 1$ for all $i$, so that

  
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the time series is regimewise-$I(0)$ under $H_{1,k}^{(0)}$. The BP test for a fixed number $m = k$ of changes is given by

$$G_1(\lambda, k) = [T - 2(k + 1)](SSR_0^{(0)} - SSR_{1,k}^{(0)})/[kSSR_{1,k}^{(0)}] \tag{5}$$

where $SSR_0^{(0)}$ denotes the SSR under $H_0^{(0)}$ and $SSR_{1,k}^{(0)}$ the unrestricted SSR. The BP test is $G_1(k) = \sup_{\lambda \in \Lambda_\epsilon^k} G_1(\lambda, k)$. To control asymptotic size when the process is either $I(1)$ or $I(0)$ under the null hypothesis, KPZ proposed a joint test. Let $H_0 = H_0^{(1)} \cup H_0^{(0)}$. The test for $H_0$ is $H(k, \eta) = \min\left[W_1(k), [cv_{w,k}(\eta)/cv_{g,k}(\eta)]G_1(k)\right]$, where $cv_{w,k}(\eta)$ and $cv_{g,k}(\eta)$ are the critical values of the statistics $W_1(k)$ and $G_1(k)$ respectively, for some significance level $\eta$. Computing $G_1(.)$ and $W_1(.)$ is done using the dynamic programming algorithms of Bai and Perron (2003) and Perron and Qu (2006) respectively.

### 3.2. Tests when the Number of Breaks is Unknown

With the number of breaks unknown up to an upper bound $A$, KPZ proposed the following test statistic to detect processes alternating between $I(1)$ and $I(0)$ regimes: $Wmax_1 = \max_{1 \le k \le A} W_1(k)$. Similarly, to detect $I(0)$-preserving changes, the BP test is $UDmax_1 = \max_{1 \le k \le A} G_1(k)$. To achieve correct size under $H_0$, KPZ also suggested the test $Hmax_1(\eta) = \min\left[Wmax_1, [cv_{w,max}(\eta)/cv_{g,max}(\eta)]UDmax_1\right]$, where, $cv_{w,max}(\eta)$ and $cv_{g,max}(\eta)$ are the critical values of $Wmax_1$ and $UDmax_1$ respectively. The decision rule is to reject $H_0$ if $H\max_1(\eta) > cv_{w,max}(\eta)$.

## 4. THE LARGE SAMPLE EFFECTS OF NON-STATIONARY VOLATILITY

We consider the large sample behavior of the KPZ and BP tests in the presence of non-stationary volatility as specified in A(3)–(4). Theorems 1 and 2 show that the null limiting distributions of the tests are not pivotal and depend on the sample path of the volatility process $g(.)$; hence the tests do not have the correct asymptotic size unless $g(.)$ is a constant. As a matter of notation, for $r \in [0, 1]$, let $\widetilde{g}(r) = (\int_0^r g(s)^2)^{1/2}$, $B_{g,1}(r) = \widetilde{g}(1)^{-1} \int_0^r g(s)\mathrm{d}B_1(s)$ and $B_{g,2}(r) = \widetilde{g}(1)^{-1} \int_0^r g(s)^2\mathrm{d}B_2(s)$. The process $B_{g,1}(s)$ is Gaussian with zero mean and variance $v(s) = \widetilde{g}(s)^2/\widetilde{g}(1)^2$ so that $B_{g,1}(.)$ is a variance-transformed Brownian motion with directing process $v$; see Davidson (1994, section 29.4) and Cavaliere (2005).

**Theorem 1.** Under Assumptions A and $H_0^{(1)}$, $F_{1d}(\lambda, k) \xrightarrow{w} F_{1d}^0(\lambda, k)$ defined by

$$\frac{1}{k+1+ij} \sum_{n=\frac{1-i}{2}}^{\frac{k+ij-i}{2}} \left[ \frac{[\{B_{g,1}^{(2n+\frac{i+1}{2})}(\lambda_{2n+\frac{i+1}{2}})\}^2 - \{B_{g,1}^{(2n+\frac{i+1}{2})}(\lambda_{2n+\frac{i-1}{2}})\}^2 - \widetilde{g}(1)^{-2}\{\widetilde{g}(\lambda_{2n+\frac{i+1}{2}})^2 - \widetilde{g}(\lambda_{2n+\frac{i-1}{2}})^2\}]^2}{4\int_{\lambda_{2n+\frac{i-1}{2}}}^{\lambda_{2n+\frac{i+1}{2}}} [B_{g,1}^{(2n+\frac{i+1}{2})}(r)]^2 \mathrm{d}r} + \frac{1}{\lambda_{2n+\frac{i+1}{2}} - \lambda_{2n+\frac{i-1}{2}}} \{B_{g,1}(\lambda_{2n+\frac{i+1}{2}}) - B_{g,1}(\lambda_{2n+\frac{i-1}{2}})\}^2 \right]$$

where $j = 1$ if $k$ is even and $j = 0$ if $k$ is odd, $i = -1$ if $d = a$ and $i = 1$ if $d = b$. Also, $F_{1d}(k) \xrightarrow{w} \sup_{\lambda \in \Lambda_\epsilon^k} F_{1d}^0(\lambda, k)$ $(d = a, b)$, $W_1(k) \xrightarrow{w} \max[F_{1a}^0(k), F_{1b}^0(k)]$, $Wmax_1 = \max_{1 \le k \le A} W_1(k) \xrightarrow{w} \max_{1 \le k \le A} \left\{\max[F_{1a}^0(k), F_{1b}^0(k)]\right\}$.

**Theorem 2.** Under Assumption A and $H_0^{(0)}$, $G_1(\lambda, k) \xrightarrow{w} G_1^0(\lambda, k)$, defined by

$$\frac{1}{k} \sum_{n=1}^{k} \left[ \frac{\left\{\lambda_n B_{g,1}(\lambda_{n+1}) - \lambda_{n+1} B_{g,1}(\lambda_n)\right\}^2}{\lambda_n \lambda_{n+1}(\lambda_{n+1} - \lambda_n)} + \frac{\left\{\widetilde{g}(\lambda_n)^2 B_{g,2}(\lambda_{n+1}) - \widetilde{g}(\lambda_{n+1})^2 B_{g,2}(\lambda_n)\right\}^2}{\widetilde{g}(\lambda_n)^2 \widetilde{g}(\lambda_{n+1})^2 \left\{\widetilde{g}(\lambda_{n+1})^2 - \widetilde{g}(\lambda_n)^2\right\}} \right]$$

Also, $G_1(k) \xrightarrow{w} \sup_{\lambda \in \Lambda_\epsilon^k} G_1^0(\lambda, k)$ and $UDmax_1 \xrightarrow{w} \max_{1 \le k \le A}\{\sup_{\lambda \in \Lambda_\epsilon^k} G_1^0(\lambda, k)\}$.

  
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The non-robustness of $G_1(1)$ to shifts in the marginal distribution of the regressors was shown in Hansen (2000). The absence of large sample invariance of the KPZ and BP tests to unconditional heteroskedasticity continues to hold for the heteroskedasticity-robust versions of these tests; see Georgiev *et al*. (2018, Remark 12). Unreported simulations did not reveal any advantage of the robust versions in the presence of unconditional heteroskedasticity. Hence, we focus on non-robust version for simplicity. Note also that under $H_0^{(0)}$, the KPZ tests diverge to positive infinity while under $H_0^{(1)}$, the BP tests have incorrect asymptotic size even when conditional homoskedasticity holds. It can be shown that these properties continue to hold under Assumptions A. Monte Carlo evidence indicates that the extent of size distortions in finite samples can be considerable (see Supplement B, Table B-1, Supporting Information).

## 5. THE WILD BOOTSTRAP VERSIONS OF THE TESTS

We now propose wild bootstrap versions of the tests and establish their asymptotic validity under Assumption A. Unlike the standard residual bootstrap, the wild bootstrap procedure (Liu, 1988) can mimic the pattern of heteroskedasticity in the errors. We also show that the bootstrap KPZ and BP test statistics are consistent under the relevant alternatives. With the direction of persistence change typically unknown, our subsequent analysis will only consider the recommended $W_1(.)$, $Wmax_1$, $G_1(.)$ and $UDmax_1$ tests. Since the null hypothesis $H_0$ involves both $I(1)$ and $I(0)$ processes, the algorithm is based on generating two kinds of bootstrap samples, one for each case, conditional on the data $\{y_t\}_{t=1}^T$. The $I(1)$ (resp., $I(0)$) bootstrap samples are used to approximate the finite sample distribution of the KPZ (resp., BP) tests. For reasons discussed below, our proposed bootstrap scheme is not recursive as in Xu (2008) for the stationary autoregressive model. Denote by $\{v_t;\ t = 1, \dots, T\}$ a sequence of i.i.d. random variables with zero mean, unit variance and uniformly bounded fourth moments (i.e., $\sup_t E(\epsilon_t^{4+\beta}) < \infty$ for some $\beta > 0$) that are independent of $\{y_t\}_{t=1}^T$. We now describe the algorithms to generate the bootstrap samples.

### 5.1. $I(1)$ Bootstrap Samples

(1) Estimate the regression $\Delta y_t = \sum_{j=1}^{l_T} \pi_j \Delta y_{t-j} + e_t^*$ $(t = l_T + 2, \dots, T)$ where $l_T$ is chosen using the BIC. Denote the estimates by $(\breve{l}_T, \breve{\pi}_1, \dots, \breve{\pi}_{\breve{l}_T})$ and construct the residuals $\breve{e}_t = \Delta y_t - \sum_{j=1}^{\breve{l}_T} \breve{\pi}_j \Delta y_{t-j}$ $(t = \breve{l}_T + 2, \dots, T)$; (2) obtain the bootstrap residuals $e_t^{(1)} = \breve{e}_t v_t$ $(t = \breve{l}_T + 2, \dots, T)$; (3) generate the bootstrap sample as follows: $y_t^{(1)} = y_{t-1}^{(1)} + e_t^{(1)}$ $(t = \breve{l}_T + 2, \dots, T)$, $y_t^{(1)} = y_t$ $(t = 1, \dots, \breve{l}_T + 1)$; (4) construct the bootstrap versions of the $W_1(.)$ and $Wmax_1$ statistics using $\{y_t^{(1)}\}_{t=1}^T$ based on a regression that *does not* include lagged first differences of $y_t^{(1)}$; 5) Repeat steps (2)–(4) $B$ times to approximate the bootstrap distribution of the statistics.

### 5.2. $I(0)$ Bootstrap Samples

The algorithm is the same except that in step (1) the regression is $y_t = c + \alpha y_{t-1} + \sum_{j=1}^{l_T} \pi_j \Delta y_{t-j} + e_t^*$ and the residuals are $\widetilde{e}_t = y_t - \widetilde{c} - \widetilde{\alpha} y_{t-1} - \sum_{j=1}^{\widetilde{l}_T} \widetilde{\pi}_j \Delta y_{t-j}$; also in step (3), we generate $y_t^{(0)} = e_t^{(0)} = \widetilde{e}_t v_t$ $(t = \widetilde{l}_T + 2, \dots, T)$, $y_t^{(0)} = 0$ $(t = 1, \dots, \widetilde{l}_T + 1)$.

For the $I(1)$ scheme, we do not introduce first-differences in step (4) to avoid explosive or multiple unit roots, as in Cavaliere and Taylor (2008a). The $I(0)$ bootstrap scheme is non-recursive since we do not 'add back' the conditional mean component based on the parameter estimates. Rather, the bootstrap samples $\{y_t^{(0)}\}$ have constant (zero) mean and are serially independent, conditional on the data. Using a recursive scheme leads to tests with lower power when the data contain an $I(1)$ segment since the estimated persistence parameter converges to 1 at rate $T$ so that the recursive bootstrap samples are effectively drawn from an autoregressive process with a root close to unity. This feature contributes to an increase in the bootstrap critical values (relative to the non-recursive

  
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bootstrap) with an adverse effect on power; see Gulesserian and Kejriwal (2014) in the context of stationarity testing based on the sieve bootstrap in the homoskedastic case. Simulations suggest notable power gains from using the non-recursive form of the wild bootstrap for alternatives that involve switches between $I(1)$ and $I(0)$ regimes (see Supplement C). Note that, in both cases, step (4) constructs the bootstrap statistics from an AR(1) specification. The bootstrap residuals are then serially independent conditional on the data; hence, no need to control for serial correlation through lagged differences. Unreported simulations showed that an AR(1) bootstrap specification resulted in improved finite sample properties (size and power).

Denote the bootstrap analogues of $W_1(k)$, $Wmax_1$, $G_1(k)$ and $UDmax_1$ by $W_1^*(k)$, $Wmax_1^*$, $G_1^*(k)$ and $UDmax_1^*$ respectively, with associated $p$-values $p^*_{k,W_1}$, $p^*_{Wmax}$, $p^*_{k,G_1}$ and $p^*_{UDmax}$, omitting the dependence on $T$ for ease of notation. Similarly, for a given significance level $\eta$, denote the bootstrap critical values by $cv^*_{w,k}(\eta)$, $cv^*_{w,max}(\eta)$, $cv^*_{g,k}(\eta)$ and $cv^*_{g,max}(\eta)$. Our proposed statistics are the 'hybrid' tests $H^*(k,\eta) = \min[W_1(k), [cv^*_{w,k}(\eta)/cv^*_{g,k}(\eta)]G_1(k)]$ and $Hmax_1^*(\eta) = \min[Wmax_1, [cv^*_{w,max}(\eta)/cv^*_{g,max}(\eta)]UDmax_1]$. The following results state that (i) the wild bootstrap versions of the tests can successfully replicate the first-order asymptotic distribution of the original tests; (ii) for a given significance level $\eta$, the statistics $H_1^*(k,\eta)$ and $Hmax_1^*(\eta)$ have asymptotic size at most $\eta$; and (iii) the test statistics are consistent when $k = m$ changes in persistence are present.

**Theorem 3.** Under Assumption A with '$\overset{w}{\to}_p$' denoting weak convergence in probability under the bootstrap measure: (a) under $H_0^{(1)}$: (i) $W_1^*(k) \overset{w}{\to}_p \max[F^0_{1a}(k), F^0_{1b}(k)]$, $Wmax_1^* \overset{w}{\to}_p \max_{1\le k\le A}\left\{\max[F^0_{1a}(k), F^0_{1b}(k)]\right\}$; (ii) $p^*_{k,W_1} \overset{w}{\to} U[0,1]$, $p^*_{Wmax} \overset{w}{\to} U[0,1]$, a uniform distribution. Under $H_0^{(0)}$, (i) $G_1^*(k) \overset{w}{\to}_p \sup_{\lambda\in\Lambda^k_\epsilon} G_1^0(\lambda,k)$, $UDmax_1^* \overset{w}{\to}_p \max_{1\le k\le A}\{\sup_{\lambda\in\Lambda^k_\epsilon} G_1^0(\lambda,k)\}$; (ii) $p^*_{k,G_1} \overset{w}{\to} U[0,1]$, $p^*_{UDmax} \overset{w}{\to} U[0,1]$; b) Under $H_0$, $\lim_{T\to\infty} P(H^*(k,\eta) > cv^*_{w,k}(\eta)) \le \eta$ and $P(Hmax_1^*(\eta) > cv^*_{w,max}(\eta)) \le \eta$; c) With $\lambda^0 \in \Lambda^m_\epsilon$, then, under $H^{(1)}_{a,m}$, $H^{(1)}_{b,m}$ and $H^{(0)}_{1,m}$, we have $p^*_{m,W_1} \overset{p}{\to} 0$, $p^*_{Wmax} \overset{p}{\to} 0$, $p^*_{m,G_1} \overset{p}{\to} 0$, $p^*_{UDmax} \overset{p}{\to} 0$.

### 5.3. Estimating the Number of Breaks

A bootstrap procedure can be devised to estimate the number of breaks based on a sequential test of $l$ versus $l+1$ breaks, following Kejriwal (2019) who assumed conditional homoskedasticity. Heteroskedasticity precludes using critical values obtained using the full sample when testing stability in each segment. Hence, we propose a new bootstrap sequential procedure. We first apply a sequential test of the null hypothesis of $l\ (\ge 1)$ breaks against the alternative of $(l+1)$ breaks. We partition the sample into $(l+1)$ segments using the $l$ estimated break dates $(\hat{T}_1, \ldots, \hat{T}_l)$ obtained by minimizing the unrestricted $SSR$. The one break KPZ and BP statistics are then applied to all estimated $(l+1)$ regimes with the statistics denoted by $W_1^{(i)}(1)$ and $G_1^{(i)}(1)$ respectively, for $i = 1, \ldots, l+1$. The parameter estimates in the $(l+1)$ regimes are used to generate the regime-specific $I(1)$ and $I(0)$ bootstrap samples, which are used to compute the bootstrap $p$-values of the statistics $W_1^{(i)}(1)$ and $G_1^{(i)}(1)$, denoted by $p^{*,(i)}_{1,W_1}$ and $p^{*,(i)}_{1,G_1}$. For a given significance level $\eta$, we reject the null of $l$ breaks in favor of $(l+1)$ breaks if $\min_{1\le i\le l+1}\{p_i^*\} < \eta_{l+1}$ (decision rule), where $p_i^* = \max\{p^{*,(i)}_{1,W_1}, p^{*,(i)}_{1,G_1}\}$ and $\eta_{l+1} = 1-(1-\eta)^{1/(l+1)}$. As shown in Supplement A, this decision rule has asymptotic size at most $\eta$ under the null hypothesis of $l$ breaks. The steps to implement the sequential procedure are the following: (i) test the null of no break ($H_0$) against the alternative of at least one break. For a given significance level $\eta$, reject $H_0$ if $p^*_{max} = \max\{p^*_{Wmax}, p^*_{UDmax}\} < \eta$ and conclude in favor of at least one break; otherwise stop and the number of breaks selected is 0; (ii) on a rejection, use the decision rule with $l = 1$ to determine if there is more than one break. Repeat by increasing $l$ sequentially until the test fails to reject the null hypothesis of no additional break; (iii) the estimate $\hat{m}$ is obtained as the total number of rejections obtained from steps 1 and 2. The probability of selecting the true number of breaks is then at least $(1-\eta)$ in large samples as stated in Theorem 1.

**Theorem 4.** Under the conditions of Theorem 3, $\lim_{T\to\infty} P(\hat{m} = m) \ge 1-\eta$.

  
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## 6. EXTENSIONS

This section discusses extensions to deal with: (i) the presence of deterministic trends; (ii) distinguishing between a pure trend shifts process from one exhibiting shifts in persistence.

### 6.1. Deterministic Trends

We consider an extension of (1) that includes the possibility of $m$ breaks in the deterministic trend, so that:

$$y_t = \mu_0 + \beta_0 t + \sum_{j=1}^{m} \mu_j DU_{jt} + \sum_{j=1}^{m} \beta_j DT_{jt} + u_t \quad (t = T_{i-1}^0 + 1, \ldots, T_i^0) \tag{6}$$

for $i = 1, \ldots, m+1$, with $u_t$ as defined by (1), where $DU_{jt} = I(t > T_j^0)$, $DT_{jt} = I(t > T_j^0)(t - T_j^0)$, $j = 1, \ldots, m$. We define $\lambda_j^0 = T_j^0/T$ and for some generic break date $\lambda_j = T_j/T$ and $\Lambda_\epsilon^m = \{(\lambda_1, \ldots, \lambda_m);\ \left|\lambda_{j+1} - \lambda_j\right| \geq \epsilon$ $(j = 1, \ldots, m-1), \lambda_1 \geq \epsilon, \lambda_m \leq 1 - \epsilon\}$, for some small $\epsilon > 0$. The DGP (and regression used) can be expressed as

$$y_t = c_i + b_i t + \alpha_i y_{t-1} + \sum_{j=1}^{p-1} \pi_{ij} \Delta y_{t-j} + e_t \tag{7}$$

with $c_i = (1 - \alpha_i)\{\mu_0 + \sum_{j=1}^{i-1}(\mu_j - \beta_j T_j^0) + u_{T_{i-1}^0}\} + (\alpha_i - \sum_{j=1}^{p-1} \pi_{ij})\{\beta_0 + \sum_{j=1}^{i-1} \beta_j\}$, $b_i = (1 - \alpha_i)(\beta_0 + \sum_{j=1}^{i-1} \beta_j)$. KPZ proposed tests of the null hypothesis $\widetilde{H}_0^{(1)}$: $c_i = c,\ \alpha_i = 1$ for all $i$ . Note that under $\widetilde{H}_0^{(1)}$, $b_i = 0$ for all $i$ so that the process follows a stable unit root process with possible drift. Again, two models are consider depending on whether the initial regime is trend or difference stationary. In accordance with the notation in Section 3, the test statistics are denoted by $F_{2a}(\lambda, k)$, $F_{2b}(\lambda, k)$, $W_2(k)$ and $Wmax_2$. The null hypothesis of a stable trend stationary process is given $\widetilde{H}_0^{(0)}$: $c_i = c,\ b_i = b,\ \alpha_i = \alpha$ for all $i$ where $|\alpha| < 1$ and the test for a fixed number of $m = k$ changes and known break dates is $G_2(\lambda, k) = [T - 3(k+1)](\widetilde{SSR}_0^{(0)} - SSR_{2,k}^{(0)})/[kSSR_{2,k}^{(0)}]$, where $\widetilde{SSR}_0^{(0)}$ denotes the SSR under $\widetilde{H}_0^{(0)}$, that is, obtained from OLS estimation of (7) subject to the restrictions $c_i = c,\ b_i = b,\ \alpha_i = \alpha$ for all $i$, and $SSR_{2,k}^{(0)}$ denotes the unrestricted SSR. Since, in general the break dates are unknown, the test statistic is defined as $G_2(k) = \sup_{\lambda \in \Lambda_\epsilon^m} G_2(\lambda, k)$. When the number of breaks is unknown, the relevant test statistic is $UDmax_2 = \max_{1 \leq k \leq A} G_2(k)$. The limit distributions of $G_2(.)$ and $UDmax_2$ under homoskedastic errors are derived in Kejriwal (2019). Under Assumption A, the above test statistics are not asymptotically pivotal and depend on the sample path of $\{\sigma_t\}$. We propose the following bootstrap algorithm that enables asymptotically valid inference. As in Section 5, we generate both $I(1)$ and $I(0)$ bootstrap samples to ensure that the procedure has correct asymptotic size under $\widetilde{H}_0 = \widetilde{H}_0^{(1)} \cup \widetilde{H}_0^{(0)}$. The algorithms are exactly the same except that for the $I(1)$ (resp., $I(0)$) case a constant (resp., a time trend) is added in the autoregression in step (1). The bootstrap analogues of $W_2(k)$, $Wmax_2$, $G_2(k)$ and $UDmax_2$ and the associated $p$-values are obtained as described in Section 5. The sequential procedure outlined in Section 5.3 is accordingly modified. The following result states the large sample validity of the proposed procedures.

**Theorem 5.** Under Assumption A, and using the tests and bootstrap procedures described above: Theorem 3 holds with $H_0$ replaced by $\widetilde{H}_0$.

### 6.2. Disentangling Trend and Persistence Shifts

An important feature is that the statistics test the null hypothesis that the persistence parameters and those of the trend function are jointly stable. Hence, they can have power against processes driven by pure trend shifts with no change in persistence. To distinguish between trend and persistence shifts, we can adapt the three-step approach of Kejriwal (2019) for the homoskedastic case to the present context. Consider first the non-trending case. The first step entails determining the number of breaks ($\widetilde{m}$) using the sequential procedure described in Section 5.3 and the associated breakpoint estimates ($\hat{T}_1, \dots, \hat{T}_{\hat{m}}$) obtained from the unrestricted model that allows all parameters including those of the lagged first differences to change across regimes. Second, using the estimated breakpoints, the Wald statistic for testing the null hypothesis of stable $I(0)$ persistence is constructed (i.e., constancy of $\alpha_i$ over all $i$) while allowing all other parameters to vary across the ($\hat{m}+1$) regimes. To account for non-stationary volatility, the Wald statistic is computed using a heteroskedasticity-robust estimator of the variance–covariance matrix (cf., Phillips and Xu, 2006). Third, the null hypothesis of stable $I(0)$ persistence is rejected if the Wald statistic is significant using the critical value from a $\chi^2(\hat{m})$ distribution. Otherwise, the null is not rejected and we conclude in favor of a model with pure level shifts.

The trending case is more complex since the process can be either $I(1)$ (with a possibly time-varying drift) or $I(0)$ (around a broken deterministic trend). As above, we develop tests separately for the $I(1)$ and $I(0)$ null and use the intersection of the critical regions of the two tests. The three-step approach is implemented as follows. First, estimate the number of breaks (say $\check{m}$) applying our proposed sequential procedure and breakpoints from the unrestricted specification. Second, compute the Wald statistic (using heteroskedasticity robust standard errors) to test the null of constant persistence allowing the parameters of the trend and lagged differences to change at the estimated breakpoints. In the $I(0)$ case, the statistic has a limiting $\chi^2(m)$, with $\check{m}$ used to obtain critical values. In the $I(1)$ case, apply a second wild bootstrap scheme based on residuals estimated under (3) allowing the constant to change across regimes at the estimated breakpoints. The $I(1)$ bootstrap samples are obtained from a DGP that now includes the estimated regime-specific drift in step (3) of the bootstrap algorithm. The bootstrap distribution and critical values of the Wald statistic can then be approximated using simulations. Finally, the null hypothesis of stable [$I(1)$ or $I(0)$] persistence is rejected if the $I(0)$ and $I(1)$ Wald statistics are both significant.

## 7. SUMMARY OF THE SIMULATIONS

This section summarizes the results of simulation experiments designed to assess the finite sample performance of our procedures and to provide a comparison with existing approaches. The full set of results is available in Supplements B–C. Following CT, we consider three specifications for the volatility process: (i) single discrete break; (ii) deterministically trending volatility; (iii) near-integrated stochastic volatility. Three types of error structures are considered: i.i.d., AR(1) and MA(1). While our theory does not formally allow for moving average processes, we nevertheless include this case in our simulations as a robustness check. The wild bootstrap is implemented using a two point distribution, that is, $v_t \in \{-1, 1\}$ with equal probability. The trimming is set at $\epsilon = .15$, $T \in \{200, 400\}$ and 1000 replications are used. We report results for the non-trending case only (those for the trending case are qualitatively similar). The lag length in the KPZ and BP procedures is selected using BIC with maximal value set to five. We report the performance of the tests $H^*(k, \eta);\ k = 1, 2$ and $Hmax_1^*(\eta) = \max\{H^*(1, \eta), H^*(2, \eta)\}$ as well as their non-robust (homoskedasticity-based) asymptotic counterparts $H(k, \eta);\ k = 1, 2$ and $Hmax_1(\eta) = \max\{H(1, \eta), H(2, \eta)\}$. The ratio-based bootstrap tests of CT are designed to test the $I(0)$ null hypothesis while our tests allow the process to be either $I(1)$ or $I(0)$ under the null. Furthermore, while our tests are based on a finite order autoregressive model, the CT tests are non-parametric and based on a mixing-type assumption for the innovations. Given that conducting a full and fair comparison of tests with different underlying models and null hypotheses is not possible, we did not include the CT tests in our analysis. The main findings are summarized as follows:

### 7.1. Finite Sample Size

The asymptotic tests are considerably oversized indicating their lack of robustness to non-stationary volatility, consistent with the large sample results in Section 4. In contrast, the proposed bootstrap tests are robust to $I(1)$ or $I(0)$ processes maintaining empirical size close to the nominal 5% level across the different volatility specifications. The same is generally true for the different error structures considered.

  
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### 7.2. Finite Sample Power

We consider DGPs with one and two breaks involving switches between $I(1)$ and $I(0)$ regimes as well as between $I(0)$-preserving regimes. In terms of size-adjusted power, the bootstrap tests are broadly comparable to their asymptotic counterparts, with neither class of tests uniformly dominating the other. The effect of underspecifying the number of breaks can be seen by comparing the power of $H^*(1)$ and $H^*(2)$ for DGPs with two breaks, where the former is generally less powerful than the latter, though not in all cases. The $Hmax_1^*$ test often has power close to that of the more powerful test amongst $H^*(1)$ and $H^*(2)$, highlighting the practical advantage of using $Hmax_1^*$ to detect the presence of at least one break. Furthermore, the proposed tests have substantial power against $I(0)$-preserving breaks, a feature that distinguishes these tests from most existing persistence change tests (e.g., the ratio-based tests) that are designed to detect switches between $I(1)$ and $I(0)$ regimes. Finally, the proposed tests are generally more powerful with deterministic rather than stochastic volatility.

### 7.3. Number of Breaks

The sequential procedure is generally reliable in selecting the number of breaks in the stable and single break cases. Its performance deteriorates in the two breaks case when the probability of underestimation can be non-negligible. For instance, with an abrupt increase in volatility, the breakpoint estimate used to partition the sample is typically close to the second true breakpoint, so that the first segment includes an $I(0)$ to $I(1)$ break while the second is $I(0)$. Whether a second break is selected depends on the power of the single break test in the first segment, which is relatively low. A notable improvement is observed as the magnitude of the volatility shift decreases and/or the shift occurs near the second persistence break in the increasing volatility case and near the first break otherwise.

### 7.4. Disentangling Trend and Persistence Shifts

We consider DGPs with a single break in persistence in addition to a DGP that involves a pure mean shift and apply the procedure proposed in Section 6 to compute the probabilities of selecting the true model for each for these DGPs. The performance is generally satisfactory and improves as $T$ increases.

## 8. EMPIRICAL APPLICATION

This section undertakes a detailed empirical examination of the nature of inflation persistence for a set of OECD countries. Our analysis sheds light on whether persistent, though stable, inflation should be regarded as a metric for evaluating macroeconomic models or if persistence varies across monetary policy regimes depending on the relative importance accorded to inflation in the monetary authority's objective function. While early empirical studies (e.g., Cogley and Sargent, 2001) on the stability of inflation persistence assumed constant volatility, subsequent work recognized the importance of allowing for time-varying volatility. A substantial literature uses Bayesian methods in a time-varying parameter VAR framework with stochastic volatility to study the stability of the persistence of the inflation-gap, defined as the deviation of inflation from its target level. These studies typically assume that the target level of inflation evolves as a driftless random walk (see, e.g., Cogley *et al*., 2010).[1] Our proposed procedures complement this literature by examining the persistence properties of inflation itself without restricting it to be $I(1)$ *apriori* as well as obviating the need to specify prior distributions. Our approach also improves on existing frequentist analyses on the topic that either assume homoskedasticity or restrict the nature of the null and alternative hypotheses allowed and thereby potentially overstate/understate the aggressiveness of the monetary policy stance towards combating inflation. For instance, Bataa *et al*. (2014) used the BP approach

[1] An exception is Eo (2016) who considers the persistence of inflation itself within a Markov-switching framework with the noise driven by normally distributed innovations.




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Table I. Break selection in OECD inflation rates

| Country (1) | $\hat{m}$ (2) | Break dates (3) | Pure mean shifts (4) | Largest AR sum (5) | 90% band (6) | Selected model ($H^*$) (7) | Selected model ($\mathcal{K}$) (8) | ADF $p$-value (9) |
|---|---|---|---|---|---|---|---|---|
| Austria | 1 | 1969:7 | Yes | 0.65 | [.47, 0.87] | $I(0)$ with one mean shift | $I(0)-I(1)$ | 0.16 |
| Belgium | 1 | 1974:9 | No | 0.82 | [0.71, 1.05] | $I(0)-I(1)$ | $I(1)-I(0)$ | 0.11 |
| Canada | 0 | – | No | 0.88 | [0.81, 1.04] | $I(1)$ | $I(1)-I(0)$ | 0.14 |
| Finland | 0 | – | No | 0.87 | [0.78, 1.06] | $I(1)$ | $I(0)-I(1)$ | 0.27 |
| France | 2 | 1973:1; 1983:4 | No | 0.65 | [0.51, 0.89] | $I(0)-I(0)-I(0)$ | $I(0)-I(1)/I(1)-I(0)$ | 0.10 |
| Germany | 2 | 1970:9; 1993:7 | No | 0.52 | [0.42, 0.64] | $I(0)-I(0)-I(0)$ | $I(0)-I(1)$ | 0.00 |
| Greece | 0 | – | No | 0.81 | [0.64, 1.12] | $I(1)$ | $I(0)-I(1)/I(1)-I(0)$ | 0.10 |
| Italy | 2 | 1972:6; 1979:10 | No | 0.88 | [0.82, 1.02] | $I(0)-I(0)-I(1)$ | $I(0)-I(1)/I(1)-I(0)$ | 0.07 |
| Japan | 0 | – | No | 0.84 | [0.66, 1.18] | $I(1)$ | $I(0)-I(1)$ | 0.31 |
| Korea | 1 | 1981:9 | Yes | 0.14 | [0.00, 0.29] | $I(0)$ with one mean shift | $I(0)-I(1)/I(1)-I(0)$ | 0.00 |
| Luxembourg | 1 | 1999:1 | No | 0.87 | [0.76, 1.10] | $I(1)-I(0)$ | $I(1)-I(0)$ | .51 |
| Netherlands | 0 | – | No | 0.04 | [−0.03, 0.12] | $I(0)$ | $I(0)$ | 0.00 |
| Norway | 0 | – | No | 0.78 | [0.66, 1.02] | $I(1)$ | $I(0)-I(1)/I(1)-I(0)$ | 0.08 |
| Portugal | 0 | – | No | 0.83 | [0.71, 1.08] | $I(1)$ | $I(0)-I(1)/I(1)-I(0)$ | 0.16 |
| Spain | 0 | – | No | 0.88 | [0.77, 1.09] | $I(1)$ | $I(0)-I(1)/I(1)-I(0)$ | 0.35 |
| Sweden | 0 | – | No | 0.82 | [0.71, 1.05] | $I(1)$ | $I(0)-I(1)$ | 0.10 |
| Switzerland | 0 | – | No | 0.82 | [0.67, 1.11] | $I(1)$ | $I(0)-I(1)$ | 0.07 |
| UK | 0 | – | No | 0.88 | [0.76, 1.11] | $I(1)$ | $I(0)-I(1)$ | 0.20 |
| USA | 0 | – | No | 0.90 | [0.83, 1.05] | $I(1)$ | $I(1)-I(0)$ | 0.16 |

This table reports the empirical results based on monthly OECD inflation rates data over 1960:1–2008:6. Column (1): the country name; column (2): the estimate $\hat{m}$ obtained from applying the sequential algorithm of Section 5.3 with $\eta = .10$ and $A = 5$; column (3): the estimated break dates obtained by minimizing the unrestricted sum of squared residuals with $\hat{m}$ breaks; column (4): the outcome of the test for the null hypothesis of pure mean shifts; column (5): the OLS estimate of the largest sum of the AR coefficients across the estimated regimes; column (6): Andrews and Guggenberger's (2014) 90% confidence band for the largest sum of the AR coefficients; column (7): the model selected by the sequential algorithm; column (8): the model selected by the CT procedure; column (9): the $p$-value of the wild bootstrap ADF test of Cavaliere and Taylor (2009).

to identify breaks in the mean, dynamics and volatility of inflation assuming it is an $I(0)$ process under the null hypothesis of stability and regime-wise $I(0)$ in the presence of breaks, thereby ruling out $I(1)$ regimes. Noriega *et al*. (2013) and Kejriwal (2019) allowed for unit roots but assumed homoskedasticity.

Our empirical investigation is based on monthly CPI inflation data for nineteen OECD countries as used in Noriega *et al*. (2013) and Kejriwal (2019), thereby facilitating a direct comparison with these studies. The data span the period 1960:1–2008:6 ($T = 582$), except for Germany and Korea for which the starting point is 1960:2. The inflation rates are seasonally unadjusted and computed as $i_t = 1200(\ln P_t - \ln P_{t-1})$, with $P_t$ the CPI at time $t$.[2] The main results are reported in Table I. The analysis proceeds in six steps. First, we apply the sequential algorithm to estimate the number of breaks $\hat{m}$ [column (2)] with $A = 5$, $\epsilon = .15$ and $\eta = .10$. Second, conditional on $\hat{m}$, the breakpoint estimates are obtained by minimizing the unrestricted SSR [column (3)]. Third, to distinguish persistence shifts from pure mean shifts, we conduct Wald tests (at the 10% level) of the null that $i_t$ is subject to $\hat{m}$ mean shifts against the alternative of $\hat{m}$ mean and persistence shifts [column (4)]. Here, a heteroskedasticity robust standard error estimate is used to construct the statistics although a wild bootstrap approach could also be used (the results were nearly identical). Fourth, based on the selected model, the largest (across regimes) estimated sum of the autoregressive parameters is computed [column (5)] along with equal-tailed 90% confidence intervals [column (6)] based on the procedure of Andrews and Guggenberger (2014, AG henceforth) uniformly valid over the stationary and non-stationary regions and robust to conditional (though not unconditional) heteroskedasticity. We use the BIC to select the number of lags within each regime with a maximum of 12 lags. Fifth, the results of the CT procedure are included to highlight the differences in terms of model selection [columns (7) and (8)]. Sixth, unit root tests allowing for non-stationary volatility (Cavaliere and Taylor, 2009) are presented as a robustness check on the model selection [column (9)].

[2] We prefer to use seasonally unadjusted rates since commonly used adjustment procedures can have adverse effects on the power of structural change tests (see Ghysels and Perron, 1993).

  
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Table II. Regime-wise estimates of the persistence parameters for the OECD countries

| Country | $\hat{m}$ | Regime | AR sum | 90% Band | ADF $p$-value |
|---|---|---|---|---|---|
| Belgium | 1 | First | 0.57 | [0.44, 0.75] | 0.00 |
| | | Second | 0.82 | [0.71, 1.05] | 0.11 |
| France | 2 | First | 0.10 | [−0.12, 0.38] | 0.00 |
| | | Second | 0.65 | [0.51, 0.89] | 0.10 |
| | | Third | 0.63 | [0.51, 0.79] | 0.00 |
| Germany | 2 | First | 0.21 | [0.10, 0.36] | 0.00 |
| | | Second | 0.52 | [0.42, 0.64] | 0.00 |
| | | Third | 0.06 | [−0.51, 1.15] | 0.08 |
| Italy | 2 | First | 0.01 | [−0.13, 0.19] | 0.00 |
| | | Second | 0.51 | [0.36, 0.73] | 0.00 |
| | | Third | 0.88 | [0.82, 1.02] | 0.07 |
| Luxembourg | 1 | First | 0.87 | [0.76, 0.1.10] | 0.51 |
| | | Second | −0.13 | [−0.80, 1.17] | 0.01 |

We now turn to a discussion of the results. Evidence of at least one break ($\hat{m} > 0$) is obtained for seven countries, of which two (Austria, Korea) favor an $I(0)$ process with a single mean shift. The AG interval estimates are consistent with the presence of at least one $I(1)$ segment in fourteen countries of which three are subject to at least one persistence break. Next, we provide a comparison between our results and those from the CT procedure. For the latter, we first apply their test $\mathcal{K}_4$ (at the 10% level with 15% trimming) designed to detect a single persistence change $[I(1) - I(0)$ or $I(0) - I(1)]$. On a rejection, the $p$-values of their $\mathcal{K}_1$ and $\mathcal{K}_1'$ tests (designed to detect the $I(0) - I(1)$ and $I(0) - I(1)$ alternatives respectively) are computed and the direction of persistence change determined by the smaller of the two $p$-values. If both $p$-values are (near)-zero, the evidence is not conclusive. Comparing columns (7)–(8) shows that the procedures agree only for Luxembourg and Netherlands and point to different models for all other countries. The CT approach is inconclusive in seven cases. Furthermore, for the eleven cases where the proposed approach decides in favor of a pure $I(1)$ process, the CT procedure suggests a break in persistence, consistent with the fact that it is designed to test the null hypothesis of a stable $I(0)$ process. In the two cases for which we select a pure mean shift process, the CT approach again points to a persistence break, again potentially explained by the non-robustness of their approach. Column (9) supplements our analysis with unit root tests applied to the regime with the largest estimate of the sum of the autoregressive coefficients based on the selected model in column (7). We report the $p$ -values of the wild bootstrap ADF test proposed by Cavaliere and Taylor (2009) which is robust to non-stationary volatility. The lag length in the ADF regression was selected using the modified AIC (MAIC) of Ng and Perron (2001) with the maximum lag set at $\lfloor 12(T/100)^{1/4} \rfloor$. The findings match the model selection outcomes in column (7) for fourteen of the nineteen countries, indicating a fair degree of consistency between the two approaches.

Table II presents the regime-wise estimates for the countries that are subject to at least one persistence break. Interestingly, for four of the five countries with persistence breaks, the first break corresponds to an increase in persistence that occurs between the early and mid-1970s, a period often described as one of 'the Great Inflation' and commonly believed to be associated with both a high level and high degree of persistence. In contrast, for France and Germany which experience two persistence breaks, the second break is associated with a persistence decline occurring in the 1980s and 1990s. To justify the importance of allowing for non-stationary volatility, Figure D-1 in Supplement D plots the volatility estimates obtained by fitting a non-parametric regression to the squared residuals obtained by estimating the model selected in column (7) of Table I. As suggested by Xu and Phillips (2008), a Gaussian kernel is used with the bandwidth chosen by cross validation, searching over bandwidths $h_i = c_i T^{-0.4}$ $(i = 1, \ldots, 4)$ with $\{c_1, \ldots, c_4\} = \{0.25, 0.4, 0.6, 0.75\}$. The estimates show considerable variation over time with different patterns across countries. While a smooth trend suggests itself for some countries (e.g., France and Norway), more irregular movements are observed for others (e.g., Belgium, UK, USA). A similar overall picture is obtained if one plots the estimated variance profile (Figure D-2) as suggested by Cavaliere and Taylor (2007) indicating that the non-stationary behavior of the sample volatility paths is a key feature of the inflation data, which if ignored, might lead to misleading inferential results.




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Finally, to evaluate the impact of non-stationary volatility on persistence change, it is useful to compare our results with the asymptotic sequential procedure of Kejriwal (2019) which assumes homoskedasticity. Using the same dataset, Kejriwal (2019) concludes in favor of a persistence change model for six additional countries (Canada, Finland, Greece, Japan, UK, USA) all of which are found to be pure $I(1)$ processes according to our analysis that accounts for non-stationary volatility. Interestingly, Kejriwal's analysis for USA suggests a shift from a high persistence $I(0)$ regime to a low persistence $I(0)$ regime, consistent with the view (e.g., Sims, 2001) that the case for unstable persistence is weakened once allowance is made for shifts in the variance of the innovations.[3] Thus, evidence for persistence shifts obtained using more restrictive methods might overstate the role of monetary policy, for example, a purported significant decline in persistence may be attributed to a more aggressive stance taken by the monetary authority towards inflation.

## 9. CONCLUSION

We proposed wild bootstrap sup-Wald tests to detect persistence change in a time series with non-stationary volatility. The set of alternative hypotheses considered include processes exhibiting switches between $I(1)$ and $I(0)$ regimes or that remain $I(0)$ in each regime. The performance of the methods suggested was shown to be reliable in finite samples and to have better properties than existing tests. An application to inflation rates further illustrates the usefulness of the proposed approach in practice.

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## DATA AVAILABILITY STATEMENT

The data are obtained from the IMF's International Financial Statistics (https://data.imf.org/) except for Germany and Korea, which were taken from the OECD Main Economic Indicators, available at http://oecd-stats.ingenta.com.

## SUPPORTING INFORMATION

Additional Supporting Information may be found online in the supporting information tab for this article.

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[3] Using the same dataset, Noriega *et al*. (2013) applies the doubly recursive test of Leybourne *et al*. (2007) to conclude in favor of a stable $I(0)$ process for 75% of the OECD countries analyzed here.

  
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# Bootstrap Procedures for Detecting Multiple Persistence Shifts in Heteroskedastic Time Series: Supplement

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**Supplement A: Proofs of the Theoretical Results**

For a $(d \times 1)$ vector $v$, $\|v\| = (\sum_{i=1}^{d} v_i^2)^{1/2}$ denotes the standard Euclidean norm while for a random variable $v$, $\|v\|_q = (E(|v|^q)^{1/q}$ denotes the $L_q$ $(q \geq 1)$ norm. For a matrix $B$, $\|B\|$ denotes the Frobenius norm, i.e., $\|B\| = \sqrt{tr(B'B)}$ and $M_B = I - P_B$, $P_B = B(B'B)^{-1}B'$. Let $P^*$ denote the bootstrap probability measure and $E^*$ the expectation with respect to $P^*$. Define the following quantities: (i) $V(r) = diag(g^2(r)I_p, g(r))$; (ii) $D_T = diag(a_T^{-2}T^{-1}, a_T^{-1}T^{-1/2})$; (iii) For $i = 1, ..., k+1$, $Z_i = (z_{T_{i-1}+1}, ..., z_{T_i})'$ where $z_t = (y_{t-1}, 1)'$ for $t = T_{i-1}+1, ..., T_i$, $Z = (z_1, ..., z_T)'$, $Y_{-1} = (y_0, ..., y_{T-1})$, $\iota_{(T\times 1)} = (1, ..., 1)'$; (iv) $\bar{z}_i = (T_i - T_{i-1})^{-1} \sum_{t=T_{i-1}+1}^{T_i} z_t$ and $\bar{z}_{i,-1} = (T_i - T_{i-1})^{-1} \sum_{t=T_{i-1}+1}^{T_i} z_{t-1}$, $\bar{z} = T^{-1} \sum_{t=1}^{T} z_t$, $\bar{z}_{-1} = T^{-1} \sum_{t=1}^{T} z_{t-1}$. As a matter of notation, we will use $\mathcal{C} = \mathcal{C}[0,1]$ to denote the space of continuous functions on $[0,1]$ and $\mathcal{D}$ the space of right continuous with left limit processes on $[0,1]$, '$\xrightarrow{p}$' to denote convergence in probability, '$\xrightarrow{w}$' to denote weak convergence in the space $\mathcal{D}$ endowed with the Skorohod metric, '$\xrightarrow{w}_p$' to denote weak convergence in probability under the bootstrap measure (Giné and Zinn, 1990), and $\lfloor . \rfloor$ to denote the integer part of its argument. Further, $B_1(.)$ and $B_2(.)$ denote standard independent Brownian motions on $[0,1]$ and $B(.) = [B_1(.), B_2(.)]'$. For any stochastic process $Z(.)$ defined over $[0,1]$, $Z^{(i)}(.)$ denotes $Z(.)$ demeaned over $[\lambda_{i-1}, \lambda_i]$, i.e., $Z^{(i)}(r) = Z(r) - (\lambda_i - \lambda_{i-1})^{-1} \int_{\lambda_{i-1}}^{\lambda_i} Z$, $r \in [\lambda_{i-1}, \lambda_i]$. Similarly, $\breve{Z}^{(i)}(.)$ denotes $Z(.)$ detrended over $[\lambda_{i-1}, \lambda_i]$, i.e., $\breve{Z}^{(i)}(r) = Z^{(i)}(r) - [\int_{\lambda_{i-1}}^{\lambda_i} rZ^{(i)} / \int_{\lambda_{i-1}}^{\lambda_i} \{r - (\lambda_i - \lambda_{i-1})^{-1} \int_{\lambda_{i-1}}^{\lambda_i} r\}^2] \times [r - (\lambda_i - \lambda_{i-1})^{-1} \int_{\lambda_{i-1}}^{\lambda_i} r]$, $r \in [\lambda_{i-1}, \lambda_i]$. Finally, for ease of presentation, all integrals of the form $\int_a^b f(r)dr$ are expressed as $\int_a^b f$. We first state two lemmas

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that will be useful in developing the proofs of the main results.

**Lemma A.1** *[Xu, 2008] Suppose $\{y_t\}$ is generated by the AR(p) model*

$$y_t = \mu + \sum_{j=1}^{p} \theta_j (y_{t-j} - \mu) + e_t$$

*where all roots of $\theta(L) = 1 - \sum_{j=1}^{p} \theta_j L^j$ are outside the unit circle and $\{e_t\}$ satisfies Assumptions A(3-4). Let $\widetilde{y}_{t-j} = y_{t-j} - \mu$, $y_{-p,t} = (\widetilde{y}_{t-1}, ..., \widetilde{y}_{t-p})'$ and $x_t = (y'_{-p,t}, 1)'$. Also, define the $[(p+1) \times (p+1)]$ matrix $\Upsilon_T = diag(T^{1/2}, ..., T^{1/2}, T^{1/2} a_T^{-1})$. Then*

*(a) $y_{-p,t} = \sum_{j=1}^{\infty} b_j e_{t-j}$ with $b_j = (\psi_{j-1}, ..., \psi_{j-p})$ if $j \geq 1$, $\psi_j = 0$ if $j < 0$, where $\theta(L)^{-1} = \sum_{j=0}^{\infty} \psi_j L^j$, $\psi_0 = 1$, $\sum_{j=0}^{\infty} j \left|\psi_j\right| < \infty$.*

*(b) $a_T^{-2} \Upsilon_T^{-1} (\sum_{t=1}^{T} x_t x_t') \Upsilon_T^{-1} \xrightarrow{p} \Psi$ where $\Psi = \begin{pmatrix} \Omega \int g^2 & 0_{(p\times 1)} \\ 0_{(1\times p)} & 1 \end{pmatrix}$ and $\Omega = \sum_{j=1}^{\infty} b_j b_j'$.*

*(c) $a_T^{-2} \Upsilon_T^{-1} \sum_{t=1}^{T} x_t e_t \xrightarrow{w} \int V dB_{p+1}$, where $B_{p+1} = (B_p', B_1)'$ with $B_p$ is a p-vector Brownian motion with covariance matrix $\Omega$ and $B_1$ is a standard Brownian motion independent of $B_p$.*

**Lemma A.2** *Suppose $\{y_t\}$ is generated by the AR(p) model with $\alpha = 1$:*

$$y_t = \alpha y_{t-1} + \sum_{j=1}^{p-1} \pi_j \Delta y_{t-j} + e_t$$

*where $\{\pi_j\}$ satisfies Assumption A(2) and $\{e_t\}$ satisfies Assumptions A(3-4). Let $e = (e_1, ..., e_T)'$, $v_t = \Delta y_t$, $w_t = (\Delta y_{t-1}, ..., \Delta y_{t-p+1})'$, $W = (w_1, ..., w_T)'$, $W_j = (w_{T_{j-1}+1}, ..., w_{T_j})'$ $(j = 1, ..., k+1)$ and $\Pi = (\pi_1, ..., \pi_{p-1})'$. Then*

*(a) $a_T^{-1} T^{-1/2} \sum_{t=1}^{\lfloor Tr \rfloor} e_t \xrightarrow{w} \int_0^r g dB_1 \equiv \widetilde{g}(1) B_{g,1}(r)$;*

*(b) $a_T^{-1} T^{-1/2} \sum_{t=1}^{\lfloor Tr \rfloor} v_t \xrightarrow{w} d(1) \int_0^r g dB_1 \equiv d(1) \widetilde{g}(1) B_{g,1}(r)$, if $d(1) \neq 0$, with $v_t = \sum_{j=0}^{\infty} d_j e_{t-j}$ and $\sum_{j=0}^{\infty} j \left|d_j\right| < \infty$;*

*(c) $a_T^{-2} T^{-1} \sum_{t=1}^{\lfloor Tr \rfloor} y_{t-1} e_t \xrightarrow{w} (1/2) d(1) [\widetilde{g}(1)^2 B_{g,1}^2(r) - \widetilde{g}(r)^2]$;*

*(d) $\left\|(a_T^{-2} T^{-1} W'W)^{-1}\right\| = O_p(1)$;*

*(e) $\|D_T Z_{2i}' W_{2i}\| = O_p(1)$;*

*(f) $\left\|a_T^{-2} T^{-1/2} W'e\right\| = O_p(1)$;*

*(g) $\left\| \left[ a_T^{-2} T^{-1} W'W - a_T^{-2} T^{-1} \sum_{i=1}^{k/2} W_{2i}' Z_{2i} (Z_{2i}' Z_{2i})^{-1} Z_{2i}' W_{2i} \right]^{-1} \right\| = O_p(1)$.*

**Proof of Lemma A.2:** (a) The result follows from Lemma 1 in Cavaliere and Taylor (2009).

(b) By Assumption A2, $\Delta y_t = v_t = \sum_{j=0}^{\infty} d_j e_{t-j}$ with $\sum_{j=0}^{\infty} j\,|d_j| < \infty$, where $d(L) = \sum_{j=0}^{\infty} d_j L^j = \pi(L)^{-1}$. Then, with the additional restriction $d(1) \neq 0$, the sequence $\{v_t\}$ satisfies Assumption 1' in Cavaliere and Taylor (2009) and hence by their Theorem 3, $a_T^{-1} T^{-1/2} \sum_{t=1}^{\lfloor Tr \rfloor} v_t \xrightarrow{w} d(1) \int_0^r g dB_1$.

(c) Note that from the Beveridge-Nelson (1981) decomposition, we have $T^{-1} \sum_{t=1}^{\lfloor Tr \rfloor} y_{t-1} e_t = d(1) T^{-1} \sum_{t=2}^{\lfloor Tr \rfloor} \{\sum_{j=1}^{t-1} e_j\} e_t + o_p(1)$. Next, using the fact that

$$T^{-1} \sum_{t=2}^{\lfloor Tr \rfloor} \{\sum_{j=1}^{t-1} e_j\} e_t = (1/2)[(T^{-1/2} \sum_{t=1}^{\lfloor Tr \rfloor} e_t)^2 - T^{-1} \sum_{t=1}^{\lfloor Tr \rfloor} e_t^2] + o_p(1)$$

the result follows from (a) since $a_T^{-2} T^{-1} \sum_{t=1}^{\lfloor Tr \rfloor} e_t^2 \xrightarrow{w} \int_0^r g(s)^2 \equiv \widetilde{g}(r)^2$.

(d) The entries in the matrix $a_T^{-2} T^{-1} W'W$ are of the form $T^{-1} \sum_{t=1}^{T} \Delta y_{t-j} \Delta y_{t-j'}$, $j, j' \in \{1, ..., p-1\}$. When $\alpha = 1$, $\{\Delta y_t\}$ is an $AR(p-1)$ process with all roots outside the unit circle. Then by Lemma A.1(b), $T^{-1} \sum_{t=1}^{T} \Delta y_{t-j} \Delta y_{t-j'} = O_p(1)$ and the result follows.

(e) We have $a_T^{-1} T^{-1/2} y_{\lfloor Tr \rfloor} = O_p(1)$ uniformly in $r \in [0,1]$. For a fixed $j \in \{1, ..., p-1\}$,

$$\begin{aligned} a_T^{-1} T^{-1/2} \sum_{t=T_{2i-1}+1}^{T_{2i}} \Delta y_{t-j} &= a_T^{-1} T^{-1/2} y_{T_{2i}-j} - a_T^{-1} T^{-1/2} y_{T_{2i-1}+1-j} \\ &\xrightarrow{w} d(1) \widetilde{g}(1) \{B_{g,1}(\lambda_{2i}) - B_{g,1}(\lambda_{2i-1})\} = O_p(1). \end{aligned}$$

Further,

$$\begin{aligned} a_T^{-2} T^{-1} \sum_{t=T_{2i-1}+1}^{T_{2i}} y_{t-1} \Delta y_{t-j} &= \sum_{t=T_{2i-1}+1}^{T_{2i}} (a_T^{-1} T^{-1/2} y_{t-1})(a_T^{-1} T^{-1/2} \Delta y_{t-j}) \\ &\xrightarrow{w} (1/2)\{d(1)^2 \widetilde{g}(1)^2 [B_{g,1}^2(\lambda_{2i}) - B_{g,1}^2(\lambda_{2i-1})] \\ &\quad -(\lambda_{2i} - \lambda_{2i-1})\xi_0\} + (\lambda_{2i} - \lambda_{2i-1}) \left\{\xi_0 + \xi_1 + ... + \xi_{j-1}\right\} = O_p(1) \end{aligned}$$

where $\xi_j = E(\Delta y_t \Delta y_{t-j}) = \widetilde{g}(1)^2 \sum_{s=0}^{\infty} d_s d_{s+j}$. Hence, all entries in the matrix $D_T Z_{2i}' W_{2i}$ are $O_p(1)$ and the result follows.

(f) The result follows by applying Lemma A.1(c) to the sequence $\{\Delta y_t\}$.

(g) First, observe that $a_T^{-2} T^{-1} W'W = O_p(1)$ by Lemma A.1(b). Next, we have

$$a_T^2 D_T Z_{2i}' Z_{2i} D_T \xrightarrow{w} \begin{bmatrix} d(1)^2 \widetilde{g}(1)^2 \int_{\lambda_{2i-1}}^{\lambda_{2i}} B_{g,1}^2 & d(1)\widetilde{g}(1) \int_{\lambda_{2i-1}}^{\lambda_{2i}} B_{g,1} \\ d(1)\widetilde{g}(1) \int_{\lambda_{2i-1}}^{\lambda_{2i}} B_{g,1} & \lambda_{2i} - \lambda_{2i-1} \end{bmatrix} \equiv \mathcal{W}_{1,i}$$

Denote the limit of $D_T Z_{2i}' W_{2i}$ by $\mathcal{W}_{2,i}$. Thus combining the results in (e),

$$\begin{aligned} &T^{-1} \sum_{i=1}^{k/2} [W_{2i}' Z_{2i} D_T] \left[(a_T^2 D_T Z_{2i}' Z_{2i} D_T)^{-1}\right] [D_T Z_{2i}' W_{2i}] \\ &\xrightarrow{w} T^{-1} \sum_{i=1}^{k/2} \mathcal{W}_{2,i}' \mathcal{W}_{1,i}^{-1} \mathcal{W}_{2,i} = O_p(T^{-1}) O_p(1) = o_p(1) \end{aligned}$$

and the result follows. ▲

**Proof of Theorem 1:** We prove the result for Model 1a and $k$ even. The proofs for the other tests are very similar and omitted. Let $\widetilde{E}_i^*$ and $\hat{E}_i^*$ be the vector of residuals in the $i$-th regime under $H_0^{(1)}$ and $H_{a,k}^{(1)}$, respectively, for $i = 1, ..., k+1$. Denote $\hat{\gamma}_{2i} = (\hat{\alpha}_{2i} - 1, \hat{c}_{2i})'$, $i = 1, ..., k/2$, where $\hat{\alpha}_{2i}$ and $\hat{c}_{2i}$ are the OLS estimates obtained from regime $2i$. Then we have

$$\begin{aligned}
\widetilde{E}_i^* &= \Delta Y_i - W_i \breve{\Pi}, && \text{for } i = 1, ..., k+1 \\
\hat{E}_{2i}^* &= \Delta Y_{2i} - W_{2i}\hat{\Pi} - Z_{2i}\hat{\gamma}_{2i}, && \text{for } i = 1, ..., k/2 \\
\hat{E}_{2i+1}^* &= \Delta Y_{2i+1} - W_{2i+1}\hat{\Pi}, && \text{for } i = 0, ..., k/2
\end{aligned} \tag{A.1}$$

where $\breve{\Pi} - \Pi = (W'W)^{-1} W'e$ under $H_0^{(1)}$. Further, $\hat{\Pi}$ and $\hat{\gamma}_{2i}$ satisfy the first order conditions

$$Z_{2i}'\hat{E}_{2i}^* = 0, \text{ for } i = 1, ..., k/2 \ , \tag{A.2}$$

$$\textstyle\sum_{i=1}^{k/2} W_{2i}\hat{E}_{2i}^* + \sum_{i=0}^{k/2} W_{2i+1}\hat{E}_{2i+1}^* = 0. \tag{A.3}$$

Under $H_0^{(1)}$, from $(A.3)$, we have $\hat{\Pi} - \Pi = (W'W)^{-1}(W'e - \sum_{i=1}^{k/2} W_{2i}'Z_{2i}\hat{\gamma}_{2i})$. Next, from $(A.2)$,

$$a_T^{-2}D_T^{-1}\hat{\gamma}_{2i} = (a_T^2 D_T Z_{2i}'Z_{2i}D_T)^{-1}\left[D_T Z_{2i}'W_{2i}(\Pi - \hat{\Pi}) + D_T Z_{2i}'E_{2i}\right] \tag{A.4}$$

for $i = 1, ..., k/2$, where $E_{2i} = (e_{T_{2i-1}+1}, ..., e_{T_{2i}})'$. Solving for $(\hat{\Pi} - \Pi)$ we obtain

$$\hat{\Pi} - \Pi = [W'W - \textstyle\sum_{i=1}^{k/2}\left\{W_{2i}'Z_{2i}(Z_{2i}'Z_{2i})^{-1}Z_{2i}'W_{2i}\right\}]^{-1}[W'e - \sum_{i=1}^{k/2}\left\{W_{2i}'Z_{2i}(Z_{2i}'Z_{2i})^{-1}Z_{2i}'E_{2i}\right\}] \tag{A.5}$$

so that using Lemma A.2, and noting that the limits of $W_{2i}'Z_{2i}D_T$, $a_T^2 D_T Z_{2i}'Z_{2i}D_T$ and $D_T Z_{2i}'E_{2i}$ are $O_p(1)$ uniformly in $i$,

$$\begin{aligned}
||\hat{\Pi} - \Pi|| &\leq \left\|[W'W - \textstyle\sum_{i=1}^{k/2}\left\{W_{2i}'Z_{2i}(Z_{2i}'Z_{2i})^{-1}Z_{2i}'W_{2i}\right\}]^{-1}\right\| \times \\
&\quad \left[\|W'e\| + \textstyle\sum_{i=1}^{k/2}\left\{\|W_{2i}'Z_{2i}D_T\|\left\|(D_T Z_{2i}'Z_{2i}D_T)^{-1}\right\|\|D_T Z_{2i}'E_{2i}\|\right\}\right] \\
&= [O_p(a_T^{-2}T^{-1})][O_p(a_T^2 T^{1/2}) + (k/2)\{O_p(1)O_p(a_T^2)O_p(1)\}] = O_p(T^{-1/2})
\end{aligned}$$

Also,

$$\begin{aligned}
||(D_T Z_{2i}'Z_{2i}D_T)^{-1}D_T Z_{2i}'W_{2i}(\Pi - \hat{\Pi})|| &\leq ||(D_T Z_{2i}'Z_{2i}D_T)^{-1}||||D_T Z_{2i}'W_{2i}||||(\Pi - \hat{\Pi})|| \\
&= O_p(a_T^2)O_p(1)O_p(T^{-1/2}) = O_p(a_T^2 T^{-1/2})
\end{aligned} \tag{A.6}$$

Using $(A.6)$ in $(A.4)$, we have

$$a_T^{-2}D_T^{-1}\hat{\gamma}_{2i} = (a_T^2 D_T Z_{2i}'Z_{2i}D_T)^{-1}D_T Z_{2i}'E_{2i} + o_p(1). \tag{A.7}$$

Next, $\hat{\Pi} - \breve{\Pi} = -(W'W)^{-1} \sum_{i=1}^{k/2} \{W_{2i}' Z_{2i} \hat{\gamma}_{2i}\}$ so that

$$\begin{aligned} ||\hat{\Pi} - \breve{\Pi}|| &\leq \left\|(W'W)^{-1}\right\| \sum_{i=1}^{k/2} \|W_{2i}' Z_{2i} D_T\| \left\|D_T^{-1} \hat{\gamma}_{2i}\right\| \\ &= O_p(a_T^{-2} T^{-1})(k/2)\{O_p(1) O_p(a_T^2)\} = O_p(T^{-1}). \end{aligned}$$

We can write, from $(A.1)$, for $i = 1, ..., k/2$, $\widetilde{E}_{2i}^* = \hat{E}_{2i}^* + Z_{2i}\hat{\gamma}_{2i} + W_{2i}(\hat{\Pi} - \breve{\Pi})$ and for $i = 0, ..., k/2$, $\widetilde{E}_{2i+1}^* = \hat{E}_{2i+1}^* + W_{2i+1}(\hat{\Pi} - \breve{\Pi})$. Thus the numerator of the $F$ statistic can be written as

$$\begin{aligned} SSR_0^{(1)} - SSR_{1a,k}^{(1)} &= \sum_{i=1}^{k/2} \{\widetilde{E}_{2i}^{*\prime} \widetilde{E}_{2i}^* - \hat{E}_{2i}^{*\prime} \hat{E}_{2i}^*\} + \sum_{i=0}^{k/2} \{\widetilde{E}_{2i+1}^{*\prime} \widetilde{E}_{2i+1}^* - \hat{E}_{2i+1}^{*\prime} \hat{E}_{2i+1}^*\} \quad \text{(A.8)} \\ &= \sum_{i=1}^{k/2} (D_T^{-1}\hat{\gamma}_{2i})'(D_T Z_{2i}' Z_{2i} D_T) D_T^{-1} \hat{\gamma}_{2i} \\ &\quad + (\hat{\Pi} - \breve{\Pi})' \sum_{i=1}^{k/2} (W_{2i}' Z_{2i} D_T)(D_T^{-1}\hat{\gamma}_{2i}) + (\hat{\Pi} - \breve{\Pi})'(W'W)(\hat{\Pi} - \breve{\Pi}) \end{aligned}$$

where

$$\begin{aligned} ||(\hat{\Pi} - \breve{\Pi})' \sum_{i=1}^{k/2} (W_{2i}' Z_{2i} D_T)(D_T^{-1}\hat{\gamma}_{2i})|| &\leq ||\hat{\Pi} - \breve{\Pi}|| \sum_{i=1}^{k/2} \|(W_{2i}' Z_{2i} D_T)\| \left\|(D_T^{-1}\hat{\gamma}_{2i})\right\| \\ &= O_p(T^{-1})(k/2)\{O_p(1) O_p(a_T^2)\} = O_p(a_T^2 T^{-1}) \end{aligned}$$

and

$$\begin{aligned} ||(\hat{\Pi} - \breve{\Pi})'(W'W)(\hat{\Pi} - \breve{\Pi})|| &\leq ||\hat{\Pi} - \breve{\Pi}|| \, \|W'W\| \, ||\hat{\Pi} - \breve{\Pi}|| \\ &= O_p(T^{-1}) O_p(a_T^2 T) O_p(T^{-1}) = O_p(a_T^2 T^{-1}) \end{aligned}$$

Then, using $(A.7)$ in $(A.8)$, we have

$$\begin{aligned} a_T^{-2}(SSR_0^{(1)} - SSR_{1a,k}^{(1)}) &= \sum_{i=1}^{k/2} \left\{E_{2i}' Z_{2i} D_T (a_T^2 D_T Z_{2i}' Z_{2i} D_T)^{-1} D_T Z_{2i}' E_{2i}\right\} + o_p(1) \\ &= \sum_{i=1}^{k/2} \left[ \frac{\{a_T^{-2} T^{-1} \sum_{t=T_{2i-1}+1}^{T_{2i}} (y_{t-1} - \bar{y}_{2i,-1}) e_t\}^2}{a_T^{-2} T^{-2} \sum_{t=T_{2i-1}+1}^{T_{2i}} (y_{t-1} - \bar{y}_{2i,-1})^2} \right. \quad \text{(A.9)} \\ &\quad \left. + \frac{T}{T_{2i} - T_{2i-1}} \{a_T^{-1} T^{-1/2} \sum_{t=T_{2i-1}+1}^{T_{2i}} e_t\}^2 \right] \end{aligned}$$

Using Lemma A.2(a,c) in $(A.9)$, we have

$$a_T^{-2}(SSR_0^{(1)} - SSR_{1a,k}^{(1)}) \xrightarrow{w} \widetilde{g}(1)^2 \sum_{i=1}^{k/2} \left[ \begin{array}{c} \frac{\left[\left\{B_{g,1}^{(2i)}(\lambda_{2i})\right\}^2 - \left\{B_{g,1}^{(2i)}(\lambda_{2i-1})\right\}^2 - \widetilde{g}(1)^{-2}\left\{\widetilde{g}(\lambda_{2i})^2 - \widetilde{g}(\lambda_{2i-1})^2\right\}\right]^2}{4\int_{\lambda_{2i-1}}^{\lambda_{2i}} \left[B_{g,1}^{(2i)}(r)\right]^2 dr} \\ + \frac{1}{\lambda_{2i} - \lambda_{2i-1}} \{B_{g,1}(\lambda_{2i}) - B_{g,1}(\lambda_{2i-1})\}^2 \end{array} \right]$$

Finally, noting that $a_T^{-2}(T-k)^{-1} SSR_{1a,k}^{(1)} \xrightarrow{p} \int_0^1 g^2 \equiv \widetilde{g}(1)^2$, the result follows.▲

**Proof of Theorem 2:** We can write

$$SSR_0^{(0)} - SSR_{1,k}^{(0)} = D^R(1, k+1) - \sum_{i=1}^{k+1} D^U(i,i)$$

where $D^U(i,j)$ $[D^U(i,j)$, resp.,$]$ is the sum of squared residuals from the unrestricted (restricted, resp.,) regression using data from segments $i$ to $j$ (inclusively). Let $Y_{(-1)1,i}$, $Z_{1,i}$, $W_{1,i}$, and $E_{1,i}$ denote the vectors or matrices containing elements of $Y_{-1}$, $Z$, $W$ and $e$, respectively, belonging to the partition from segment 1 to $i$ (inclusively), for $i = 1, ..., k+1$. Further, define $S_i = Z'_{1,i}E_{1,i}$, $H_i = Z'_{1,i}Z_{1,i}$, $K_i = Z'_{1,i}W_{1,i}$, $L_i = W'_{1,i}W_{1,i}$ and $M_j = W'_{1,i}E_{1,i}$ for $i = 1, ..., k{+}1$. Finally, let $A_T = (W'M_ZW)^{-1}W'M_Ze$ and $\bar{A}_T = (W'M_{\bar{Z}}W)^{-1}W'M_{\bar{Z}}e$, where $\bar{Z} = diag(Z_1, ..., Z_{k+1})$. Then, from Bai and Perron (1998, eq. (39) and (41), pg. 73-74),

$$SSR_0^{(0)} - SSR_{1,k}^{(0)} = \textstyle\sum_{i=1}^{k} F_{T,i} + D^R(1,1) - D^U(1,1)$$

where

$$\begin{aligned} F_{T,i} &= \left[-S'_{i+1}H_{i+1}^{-1}S_{i+1} + S'_iH_i^{-1}S_i + (S_{i+1}-S_i)[H_{i+1}-H_i]^{-1}(S_{i+1}-S_i)\right] \\ &\quad + \left[2S'_{i+1}H_{i+1}^{-1}K_{i+1}A_T - 2S'_iH_i^{-1}K_iA_T - 2(S_{i+1}-S_i)'[H_{i+1}-H_i]^{-1}(K_{i+1}-K_i)\bar{A}_T\right] \\ &\quad + \left[2(M_{i+1}-M_i)'(\bar{A}_T - A_T) + (\bar{A}_T - A_T)'(L_{i+1}-L_i)(\bar{A}_T - A_T)\right] \\ &= T1 + T2 + T3 \end{aligned} \tag{A.10}$$

We now analyze each of the terms $T1$-$T3$ in $(A.10)$. For $T1$:

$$\begin{aligned} T1 &= -S'_{i+1}H_{i+1}^{-1}S_{i+1} + S'_iH_i^{-1}S_i + (S_{i+1}-S_i)'[H_{i+1}-H_i]^{-1}(S_{i+1}-S_i) \\ &= -\lambda_{i+1}^{-1}[\{T^{-1/2}\textstyle\sum_{t=1}^{T_{i+1}} e_t\}^2 + \{T^{-1}\sum_{t=1}^{T_{i+1}} \widetilde{y}_{t-1}^2\}^{-1}\{T^{-1/2}\sum_{t=1}^{T_{i+1}} \widetilde{y}_{t-1}e_t\}^2] \\ &\quad +\lambda_i^{-1}[\{T^{-1/2}\textstyle\sum_{t=1}^{T_i} e_t\}^2 + \{T^{-1}\sum_{t=1}^{T_i} \widetilde{y}_{t-1}^2\}^{-1}\{T^{-1/2}\sum_{t=1}^{T_i} \widetilde{y}_{t-1}e_t\}^2] \\ &\quad +(\lambda_{i+1}-\lambda_i)^{-1}[\{T^{-1/2}\textstyle\sum_{t=T_i+1}^{T_{i+1}} e_t\}^2 + \{T^{-1}\sum_{t=T_i+1}^{T_{i+1}} \widetilde{y}_{t-1}^2\}^{-1}\{T^{-1/2}\sum_{t=T_i+1}^{T_{i+1}} \widetilde{y}_{t-1}e_t\}^2] \\ &\quad +o_p(a_T^2) \end{aligned}$$

using Lemma A.1(b) where $\widetilde{y}_{t-j} = y_{t-j} - \mu$. Then, from Lemma A.1, we have

$$\begin{aligned} & a_T^{-2}T1 \\ & \xrightarrow{w} \widetilde{g}(1)^2\left[-\lambda_{i+1}^{-1}B_{g,1}^2(\lambda_{i+1}) + \lambda_i^{-1}B_{g,1}^2(\lambda_i) + (\lambda_{i+1}-\lambda_i)^{-1}[B_{g,1}(\lambda_{i+1}) - B_{g,1}(\lambda_i)]^2\right] \\ & +\widetilde{g}(1)^2\left[\begin{array}{c} -\{\widetilde{g}^2(\lambda_{i+1})\}^{-1}B_{g,2}^2(\lambda_{i+1}) + \{\widetilde{g}^2(\lambda_i)\}^{-1}B_{g,2}^2(\lambda_i) + \\ \{\widetilde{g}^2(\lambda_{i+1}) - \widetilde{g}^2(\lambda_i)\}^{-1}[B_{g,2}(\lambda_{i+1}) - B_{g,2}(\lambda_i)]^2 \end{array}\right] \\ \equiv\ & \widetilde{g}(1)^2\left[\frac{\{\lambda_iB_{g,1}(\lambda_{i+1}) - \lambda_{i+1}B_{g,1}(\lambda_i)\}^2}{\lambda_i\lambda_{i+1}(\lambda_{i+1}-\lambda_i)} + \frac{\{\widetilde{g}(\lambda_i)^2B_{g,2}(\lambda_{i+1}) - \widetilde{g}(\lambda_{i+1})^2B_{g,2}(\lambda_i)\}^2}{\widetilde{g}(\lambda_i)^2\widetilde{g}(\lambda_{i+1})^2\{\widetilde{g}(\lambda_{i+1})^2 - \widetilde{g}(\lambda_i)^2\}}\right]. \end{aligned}$$

For $T2$:

$$\begin{aligned} T2 &= 2(T^{-1/2}S_{i+1})'(T^{-1}H_{i+1})^{-1}T^{-1}K_{i+1}T^{1/2}A_T - 2(T^{-1/2}S_i)'(T^{-1}H_i)^{-1}T^{-1}K_iT^{1/2}A_T \\ &\quad -2[T^{-1/2}(S_{i+1}-S_i)]'[T^{-1}(H_{i+1}-H_i)]^{-1}T^{-1}(K_{i+1}-K_i)T^{1/2}\bar{A}_T. \end{aligned}$$

Define $\widetilde{\Omega}_{p-1} = (\Omega_{11} - \Omega_{12}, \Omega_{12} - \Omega_{13}, ..., \Omega_{1(p-1)} - \Omega_{1p})'$, where $\Omega_{ij}$ is the $(i,j)$ element of $\Omega$ defined in Lemma A.1. Then, using Lemma A.1(a)-(c), we have

$$\begin{aligned}
&a_T^{-2}(T^{-1/2}S_{i+1})'(T^{-1}H_{i+1})^{-1}T^{-1}K_{i+1} \xrightarrow{w} (1/\Omega_{11})^{1/2}B_{g,2}(\lambda_{i+1})\widetilde{\Omega}'_{p-1},\\
&a_T^{-2}(T^{-1/2}S_i)'(T^{-1}H_i)^{-1}T^{-1}K_i \xrightarrow{w} (1/\Omega_{11})^{1/2}B_{g,2}(\lambda_i)\widetilde{\Omega}'_{p-1},\\
&a_T^{-2}[T^{-1/2}(S_{i+1} - S_i)]'[T^{-1}(H_{i+1} - H_i)]^{-1}T^{-1}(K_{i+1} - K_i)\\
&\xrightarrow{w} (1/\Omega_{11})^{1/2}[B_{g,2}(\lambda_{i+1}) - B_{g,2}(\lambda_i)]\widetilde{\Omega}'_{p-1}.
\end{aligned}$$

Using Lemma A.1, it can further be shown that

$$\begin{aligned}
&T^{-1}W'P_Z W \xrightarrow{p} \Omega_{11}^{-1}\widetilde{\Omega}_{p-1}\widetilde{\Omega}'_{p-1}\widetilde{g}^2(1), \quad T^{-1}W'P_{\bar{Z}}W \xrightarrow{p} \Omega_{11}^{-1}\widetilde{\Omega}_{p-1}\widetilde{\Omega}'_{p-1}\widetilde{g}^2(1),\\
&T^{-1/2}W'P_Z e \xrightarrow{w} (1/\Omega_{11})^{1/2}\widetilde{\Omega}_{p-1}B_{g,2}(1), \quad T^{-1/2}W'P_{\bar{Z}}e \xrightarrow{w} (1/\Omega_{11})^{1/2}\widetilde{\Omega}_{p-1}B_{g,2}(1),
\end{aligned}$$

so that $\bar{A}_T - A_T \xrightarrow{p} 0$. Hence, $a_T^{-2}T2 = o_p(1)$. For $T3$:

$$a_T^{-2}T3 = a_T^{-2}[2(M_{i+1} - M_i)'(\bar{A}_T - A_T) + (\bar{A}_T - A_T)'(L_{i+1} - L_i)(\bar{A}_T - A_T)] \xrightarrow{p} 0$$

since $\bar{A}_T - A_T \xrightarrow{p} 0$. From $(A.10)$, we then obtain

$$a_T^{-2}F_{T,i} \xrightarrow{w} \widetilde{g}(1)^2\left[\frac{\{\lambda_i B_{g,1}(\lambda_{i+1}) - \lambda_{i+1}B_{g,1}(\lambda_i)\}^2}{\lambda_i\lambda_{i+1}(\lambda_{i+1} - \lambda_i)} + \frac{\{\widetilde{g}(\lambda_i)^2 B_{g,2}(\lambda_{i+1}) - \widetilde{g}(\lambda_{i+1})^2 B_{g,2}(\lambda_i)\}^2}{\widetilde{g}(\lambda_i)^2\widetilde{g}(\lambda_{i+1})^2\left\{\widetilde{g}(\lambda_{i+1})^2 - \widetilde{g}(\lambda_i)^2\right\}}\right]$$

The result follows noting that $[T - 2(k+1)]^{-1}a_T^{-2}SSR_{1,k}^{(0)} \xrightarrow{p} \widetilde{g}(1)^2$.▲

**Proof of Theorem 3(a):** We will prove the theorem for the bootstrap test based on $F_{1a}(\lambda, k)$ for $k$ even. The bootstrap statistic is given by

$$F_{1a}^*(\lambda, k) = (T-k)(SSR_0^{*,(1)} - SSR_{1a,k}^{*,(1)})/[kSSR_{1a,k}^{*,(1)}]$$

where

$$\begin{aligned}
SSR_0^{*,(1)} &= \textstyle\sum_{t=1}^T (y_t^{(1)} - y_{t-1}^{(1)})^2 && \text{(A.11)}\\
SSR_{1a,k}^{*,(1)} &= \textstyle\sum_{i=1}^{k/2}\sum_{t=T_{2i-1}+1}^{T_{2i}}(y_t^{(1)} - \bar{y}_{2i}^{(1)} - \hat{\alpha}_{2i}^{(1)}(y_{t-1}^{(1)} - \bar{y}_{2i,-1}^{(1)}))^2 && \text{(A.12)}\\
&\quad + \textstyle\sum_{i=0}^{k/2}\sum_{t=T_{2i}+1}^{T_{2i+1}}(y_t^{(1)} - y_{t-1}^{(1)})^2
\end{aligned}$$

In $(A.12)$, $\hat{\alpha}_{2i}^{(1)}$ denotes the slope estimate from an OLS regression of $y_t^{(1)}$ on a constant and $y_{t-1}^{(1)}$ $(t = T_{2i-1}+1, ..., T_{2i};\ i = 1, ..., k/2)$. Since $y_t^{(1)} = y_{t-1}^{(1)} + e_t^{(1)}$ for $t \le T$, we have

$$\begin{aligned}
&a_T^{-2}(SSR_0^{*,(1)} - SSR_{1a,k}^{*,(1)})\\
&= \sum_{i=1}^{k/2}\left[(T_{2i} - T_{2i-1})[a_T^{-1}\bar{e}_{2i}^{(1)}]^2 + \frac{[a_T^{-2}T^{-1}\sum_{t=T_{2i-1}+1}^{T_{2i}}\{(y_{t-1}^{(1)} - \bar{y}_{2i,-1}^{(1)})e_t^{(1)}\}]^2}{a_T^{-2}T^{-2}\sum_{t=T_{2i-1}+1}^{T_{2i}}(y_{t-1}^{(1)} - \bar{y}_{2i,-1}^{(1)})^2}\right] \quad \text{(A.13)}
\end{aligned}$$

Next, we establish an invariance principle for the sequence $\{a_T^{-1}e_t^{(1)}; t=1,...,T\}$. To this end, let $\mathcal{F}_t^*$ be the $\sigma$-field generated by $\{v_s; s\leq t\}$. Since $e_t^{(1)} = \breve{e}_t v_t$, $\{a_T^{-1}e_t^{(1)}, \mathcal{F}_t^*\}$ is a martingale difference array. Further, uniformly over $r\in[0,1]$,

$$a_T^{-2}T^{-1}\textstyle\sum_{t=1}^{\lfloor Tr\rfloor}[e_t^{(1)}]^2 - a_T^{-2}T^{-1}\sum_{t=1}^{\lfloor Tr\rfloor}\breve{e}_t^2 \xrightarrow{p^*} 0$$

since

$$\begin{aligned} E^*\{a_T^{-2}T^{-1}\textstyle\sum_{t=1}^{\lfloor Tr\rfloor}([e_t^{(1)}]^2-\breve{e}_t^2)\}^2 &= E^*\{a_T^{-2}T^{-1}\textstyle\sum_{t=1}^{\lfloor Tr\rfloor}\left(\breve{e}_t^2(v_t^2-1)\right)\}^2 \\ &\leq CT^{-2}\textstyle\sum_{t=1}^{\lfloor Tr\rfloor}(a_T^{-1}\breve{e}_t^2)^4 = o_p(1), \end{aligned}$$

(where $C$ is a positive constant), using the fact that under $H_0^{(1)}$, $a_T^{-1}\breve{e}_t = a_T^{-1}e_t + a_T^{-1}w_t'(\Pi - \breve{\Pi}) = a_T^{-1}e_t + O_p(T^{-1/2})$. Also, $a_T^{-2}T^{-1}\sum_{t=1t}^{\lfloor Tr\rfloor}\breve{e}_t^2 \xrightarrow{p} \int_0^r g^2$ uniformly over $r\in[0,1]$ (by Lemma 2 in Cavaliere and Taylor, 2008(a)). Then, applying Theorem 2.1 in Hansen (1992) with $S_T(.) = T^{-1/2}\sum_{t=1}^{[T.]}v_t$, we obtain

$$T^{-1/2}\textstyle\sum_{t=1}^{\lfloor Tr\rfloor}a_T^{-1}e_t^{(1)} = a_T^{-1}\int_0^r \breve{e}_{\lfloor Ts\rfloor}dS_T(s) \xrightarrow{w}_p \int_0^r g(s)dB_1(s) = \widetilde{g}(1)B_{g,1}(r). \qquad \text{(A.14)}$$

Using $(A.14)$, we have

$$\begin{aligned} &a_T^{-2}T^{-2}\sum_{t=T_{2i-1}+1}^{T_{2i}}(y_{t-1}^{(1)}-\bar{y}_{2i,-1}^{(1)})^2 \xrightarrow{w}_p \widetilde{g}(1)^2\int_{\lambda_{2i-1}}^{\lambda_{2i}}[B_{g,1}^{(2i)}(s)]^2 \\ &a_T^{-2}T^{-1}\sum_{t=T_{2i-1}+1}^{T_{2i}}\{(y_{t-1}^{(1)}-\bar{y}_{2i,-1}^{(1)})e_t^{(1)}\} \xrightarrow{w}_p (1/2)\widetilde{g}(1)^2\left[\begin{array}{c}\{B_{g,1}^{(2i)}(\lambda_{2i})\}^2-\{B_{g,1}^{(2i)}(\lambda_{2i-1})\}^2 \\ -\widetilde{g}(1)^{-2}\{\widetilde{g}(\lambda_{2i})^2-\widetilde{g}(\lambda_{2i-1})^2\}\end{array}\right] \\ &a_T^{-1}T^{1/2}\bar{e}_{2i}^{(1)} \xrightarrow{w}_p (\lambda_{2i}-\lambda_{2i-1})^{-1}\widetilde{g}(1)[B_{g,1}(\lambda_{2i})-B_{g,1}(\lambda_{2i-1})] \qquad \text{(A.15)} \end{aligned}$$

Substituting $(A.15)$ in $(A.13)$ and noting that $(T-k)^{-1}a_T^{-2}SSR_{1a,k}^{*,(1)} \xrightarrow{p} \widetilde{g}(1)^2$, $F_{1a}^*(\lambda,k) \xrightarrow{w}_p F_{1a}^0(\lambda,k)$, where $F_{1a}^0(\lambda,k)$ is the weak limit of $F_{1a}(\lambda,k)$ as stated in Theorem 1. The rest of the proof follows from the proof of Theorem 5 in Hansen (2000). The bootstrap BP test for $k$ breaks is given by

$$G_1^*(k) = [T-2(k+1)](SSR_0^{*,(0)} - SSR_{1,k}^{*,(0)})/[kSSR_{1,k}^{*,(0)}]$$

where

$$\begin{aligned} SSR_0^{*,(0)} &= \textstyle\sum_{t=1}^{T}(e_t^{(0)}-\bar{e}^{(0)}-\widetilde{\alpha}^{(0)}(e_{t-1}^{(0)}-\bar{e}_{-1}^{(0)}))^2 && \text{(A.16)} \\ SSR_{1,k}^{*,(0)} &= \textstyle\sum_{i=1}^{k+1}\sum_{t=T_{i-1}+1}^{T_i}(e_t^{(0)}-\bar{e}_i^{(0)}-\hat{\alpha}_i^{(0)}(e_{t-1}^{(0)}-\bar{e}_{i,-1}^{(0)}))^2 && \text{(A.17)} \end{aligned}$$

In $(A.16)$ (resp., $A.17$), $\widetilde{\alpha}^{(0)}$ (resp., $\hat{\alpha}_i^{(0)}$) denotes the slope estimate from an OLS regression of $e_t^{(0)}$ on a constant and $e_{t-1}^{(0)}$ $(t=1,...,T)$ (resp., $e_{t-1}^{(0)}$ $(t=T_{i-1}+1,...,T_i)$). After some algebra, we have

$$a_T^{-2}(SSR_0^{*,(0)} - SSR_{1,k}^{*,(0)}) = -T\left[a_T^{-1}\bar{e}^{(0)}\right]^2 - \frac{[a_T^{-2}T^{-1/2}\sum_{t=1}^{T}\{(e_{t-1}^{(0)}-\bar{e}_{-1}^{(0)})e_t^{(0)}\}]^2}{a_T^{-2}T^{-1}\sum_{t=1}^{T}(e_{t-1}^{(0)}-\bar{e}_{-1}^{(0)})^2}$$

$$+\sum_{i=1}^{k+1}\left[(T_i-T_{i-1})[a_T^{-1}\bar{e}_i^{(0)}]^2+\frac{[a_T^{-2}T^{-1/2}\sum_{t=T_{i-1}+1}^{T_i}\{(e_{t-1}^{(0)}-\bar{e}_{i,-1}^{(0)})e_t^{(0)}\}]^2}{a_T^{-2}T^{-1}\sum_{t=T_{i-1}+1}^{T_i}(e_{t-1}^{(0)}-\bar{e}_{i,-1}^{(0)})^2}\right]. \tag{A.18}$$

Next, we establish an invariance principle for the sequence $\{a_T^{-1}e_t^{(0)}; t=1,...,T\}$. In particular, we will show that for $r\in[0,1]$,

$$T^{-1/2}\textstyle\sum_{t=1}^{\lfloor Tr\rfloor}a_T^{-1}e_t^{(0)}\overset{w}{\to}_p\int_0^r g(s)dB_1(s). \tag{A.19}$$

To this end, let $\mathcal{F}_t^*$ be the $\sigma$-field generated by $\{v_s; s\le t\}$. Since $e_t^{(0)}=\widetilde{e}_t v_t$, $\{\ a_T^{-1}e_t^{(0)},\mathcal{F}_t^*\}$ is a martingale difference array. Further, uniformly over $r\in[0,1]$,

$$a_T^{-2}T^{-1}\textstyle\sum_{t=1}^{\lfloor Tr\rfloor}[e_t^{(0)}]^2-a_T^{-2}T^{-1}\sum_{t=1}^{\lfloor Tr\rfloor}\widetilde{e}_t^2\overset{p^*}{\to}0$$

since

$$\begin{aligned}E^*\{a_T^{-2}T^{-1}\textstyle\sum_{t=1}^{\lfloor Tr\rfloor}([e_t^{(0)}]^2-\widetilde{e}_t^2)\}^2 &= E^*\{a_T^{-2}T^{-1}\textstyle\sum_{t=1}^{\lfloor Tr\rfloor}\left(\widetilde{e}_t^2(v_t^2-1)\right)\}^2\\ &\le CT^{-2}\textstyle\sum_{t=1}^{\lfloor Tr\rfloor}(a_T^{-1}\widetilde{e}_t)^4=o_p(1)\end{aligned}$$

using the fact that $a_T^{-1}\widetilde{e}_t=a_T^{-1}e_t+O_p(T^{-1/2})$ [eq. (A.7) in Xu, 2008]. Also, $a_T^{-2}T^{-1}\sum_{t=1}^{\lfloor Tr\rfloor}\widetilde{e}_t^2\overset{p}{\to}\int_0^r g^2$. Then, again applying Theorem 2.1 in Hansen (1992) with $S_T(.)=T^{-1/2}\sum_{t=1}^{[T.]}v_t$, we have

$$T^{-1/2}\textstyle\sum_{t=1}^{\lfloor Tr\rfloor}a_T^{-1}e_t^{(0)}=a_T^{-1}\int_0^r\widetilde{e}_{\lfloor Ts\rfloor}dS_T(s)\overset{w}{\to}_p\int_0^r g(s)dB_1(s).$$

Noting that $\{a_T^{-2}e_t^{(0)}e_{t-1}^{(0)},\mathcal{F}_t^*\}$ is a martingale difference array, we can show, using similar arguments as above, that

$$T^{-1/2}\sum_{t=1}^{\lfloor Tr\rfloor}a_T^{-2}e_t^{(0)}e_{t-1}^{(0)}\overset{w}{\to}_p\int_0^r g^2(s)dB_2(s) \tag{A.20}$$

for $r\in[0,1]$, where $B_2(.)$ is independent of $B_1(.)$. Finally, since $a_T^{-2}T^{-1}SSR_{1,k}^{*,(0)}\overset{p}{\to}\widetilde{g}(1)^2$, $G_1^*(\lambda,k)\overset{w}{\to}_p G_1^0(\lambda,k)$ using $(A.19)$ and $(A.20)$ in $(A.18)$, where $G_1^0(\lambda,k)$ is the weak limit of $G_1(\lambda,k)$ as defined in Theorem 2. Hence, following the proof of Theorem 5 in Hansen (2000), $p_{k,G_1}^*\overset{w}{\to}U[0,1]$, $p_{UDmax}^*\overset{w}{\to}U[0,1]$.▲

**Proof of Theorem 3(b)**: The proof of this result follows directly from part (a) and is hence omitted.

**Proof of Theorem 3(c):** We will prove $p_{m,W_1}^*\overset{p}{\to}0$ and $p_{m,G_1}^*\overset{p}{\to}0$ under $H_{a,m}^{(1)}$ with $m$ even. Consequently, $p_{Wmax}^*\overset{p}{\to}0$ and $p_{UDmax}^*\overset{p}{\to}0$. The proofs for the alternatives $H_{b,m}^{(1)}$ and $H_{1,m}^{(0)}$ can be established using similar arguments. The proof proceeds in two steps: (i) we first show that the bootstrap counterparts $F_{1a}^*(m)$, $F_{1b}^*(m)$ and $G_1^*(m)$ of $F_{1a}(m)$, $F_{1b}(m)$ and

$G_1(m)$, respectively, are each $O_p(1)$ under $H_{a,m}^{(1)}$; (ii) $F_{1a}(m)$ (hence $W_1(m)$) and $G_1(m)$ both diverge with $T$. For (i), first note that for $s \in [0,1]$,

$$a_T^{-1}\breve{e}_{\lfloor Ts \rfloor} = a_T^{-1}e_{\lfloor Ts \rfloor} + a_T^{-1}\sum_{i=1}^{m/2}(\alpha_{2i}-1)h_{\lfloor Ts \rfloor - 1}I(\lfloor Ts \rfloor \in [T_{2i-1}^0+1, T_{2i}^0]) + O_p(T^{-1/2})$$

so that

$$a_T^{-2}T^{-1}\sum_{t=1}^{\lfloor Tr \rfloor}\breve{e}_t^2 = a_T^{-2}T^{-1}\sum_{t=1}^{\lfloor Tr \rfloor}e_t^2 + a_T^{-2}T^{-1}\sum_{t=1}^{\lfloor Tr \rfloor}\sum_{i=1}^{m/2}(\alpha_{2i}-1)^2h_{t-1}^2I(t \in [T_{2i-1}^0+1, T_{2i}^0]) + o_p(1)$$

$$\xrightarrow{p} \int_0^r g^2 + \sum_{i=1}^{m/2}(\alpha_{2i}-1)^2(\Omega_{11}^{(2i)})\int_0^r g(s)^2 I(s \in [\lambda_{2i-1}^0, \lambda_{2i}^0])ds \equiv \breve{V}(r) \tag{A.21}$$

where $a_T^{-2}T^{-1}\sum_{t=T_{2i-1}^0+1}^{\lfloor Ts \rfloor}h_{t-1}^2 \xrightarrow{p} \Omega_{11}^{(2i)}\int_{\lambda_{2i-1}^0}^s g^2$ if $s \in [\lambda_{2i-1}^0, \lambda_{2i}^0]$ and $\Omega_{11}^{(2i)}$ is the $(1,1)$ element of $\Omega^{(2i)}$ with $\Omega^{(2i)}$ defined analogously to $\Omega$ in Lemma A.1 but now specific to regime $2i$. Therefore, we have for $r \in [0,1]$,

$$a_T^{-1}T^{-1/2}\sum_{t=1}^{\lfloor Tr \rfloor}e_t^{(1)} = a_T^{-1}\int_0^r \breve{e}_{\lfloor Ts \rfloor}dS_T(s) \xrightarrow{w}_p \int_0^r g_1(s)dB_1(s) \equiv \breve{B}_{g,1}(r) \tag{A.22}$$

where $g_1(s) = g(s)[1 + \sum_{i=1}^{m/2}(\alpha_{2i}-1)^2(\Omega_{11}^{(2i)})I(s \in [\lambda_{2i-1}^0, \lambda_{2i}^0])ds]^{1/2}$. Note that $\int_0^r g_1(s)^2 = \breve{V}(r)$. Then, the results stated in $(A.15)$ all hold with $B_{g,1}(.)$ replaced by $\breve{B}_{g,1}(.)$. Further, $(T-k)^{-1}SSR_{1a,k}^{*,(1)} \xrightarrow{w}_p \int_0^1 g_1(s)^2 = \breve{V}(1)$. Thus, $F_{1a}^*(m) = O_p(1)$. Entirely analogous arguments can be used to establish $F_{1b}^*(m) = O_p(1)$. Next, we show that $G_1^*(m)$ is stochastically bounded under $H_{a,m}^{(1)}$. First, note that we can write

$$\widetilde{e}_t = y_t - \bar{y} - \widetilde{\alpha}(y_{t-1} - \bar{y}_{-1}) - (w_t - \bar{w})'\widetilde{\Pi}$$

where $T(\widetilde{\alpha}-1) = O_p(1)$ since $H_{a,m}^{(1)}$ involves a mix of $I(1)$ and $I(0)$ regimes. Further,

$$\begin{aligned} \bar{y} - \widetilde{\alpha}\bar{y}_{-1} &= \bar{y} - \bar{y}_{-1} - (\widetilde{\alpha}-1)\bar{y}_{-1} = T^{-1}(y_T - y_0) - (\widetilde{\alpha}-1)\bar{y}_{-1} \\ &= O_p(a_T T^{-1/2}) - O_p(a_T T^{-1/2}) = O_p(a_T T^{-1/2}). \end{aligned}$$

Thus, in an $I(1)$ regime, i.e., $t \in [T_{2i}+1, ..., T_{2i+1}]$, $i = 0, ..., m/2$, we have

$$\begin{aligned} a_T^{-1}\widetilde{e}_t &= a_T^{-1}e_t + a_T^{-1}(1-\widetilde{\alpha})y_{t-1} + O_p(T^{-1/2}) = a_T^{-1}e_t + O_p(T^{-1})O_p(T^{1/2}) + O_p(T^{-1/2}) \\ &= a_T^{-1}e_t + O_p(T^{-1/2}). \end{aligned} \tag{A.23}$$

In an $I(0)$ regime, i.e., $t \in [T_{2i-1}+1, ..., T_{2i}]$, $i = 1, ..., m/2$, we have

$$a_T^{-1}\widetilde{e}_t = a_T^{-1}e_t + (\alpha_{2i}-1)a_T^{-1}h_{t-1} + O_p(T^{-1/2}). \tag{A.24}$$

Combining $(A.23)$ and $(A.24)$, we have for $t = 1, ..., T$,

$$a_T^{-1}\widetilde{e}_t = a_T^{-1}e_t + \sum_{i=1}^{m/2}(\alpha_{2i}-1)a_T^{-1}h_{t-1}I(t \in [T_{2i-1}^0+1, ..., T_{2i}^0])$$

so that for $r \in [0,1]$,

$$a_T^{-2}T^{-1}\sum_{t=1}^{\lfloor Tr\rfloor}\widetilde{e}_t^2 = a_T^{-2}T^{-1}\sum_{t=1}^{\lfloor Tr\rfloor}e_t^2$$
$$+a_T^{-2}T^{-1}\sum_{t=1}^{\lfloor Tr\rfloor}\sum_{i=1}^{m/2}(\alpha_{2i}-1)^2h_{t-1}^2 I(t\in[T_{2i-1}^0+1,T_{2i}^0])+o_p(1)\xrightarrow{p}\breve{V}(r)$$

where $\breve{V}(r)$ is defined in $(A.21)$. Hence, $a_T^{-1}T^{-1/2}\sum_{t=1}^{\lfloor Tr\rfloor}e_t^{(0)} = a_T^{-1}\int_0^r \widetilde{e}_{\lfloor Ts\rfloor}dS_T(s)\xrightarrow{w}_p \breve{B}_{g,1}(r)$ and the limits in $(A.19)$ and $(A.20)$ now hold with $g(.)$ replaced by $g_1(.)$. Also, $a_T^{-2}T^{-1}SSR_{1,k}^{*,(0)}$ $\xrightarrow{p}\breve{V}(1)$. Thus, $G_1^*(m)=O_p(1)$.

To show step (ii), note that since $\lambda^0\in\Lambda_\epsilon^m$ and $F_{1a}(m)=\sup_{\lambda\in\Lambda_\epsilon^m}F_{1a}(\lambda,m)$, it is sufficient to show that $F_{1a}(\lambda^0,m)=O_p(T)$. Define

$$\begin{aligned}\breve{\Pi} &= (\textstyle\sum_{t=1}^T w_tw_t')^{-1}\sum_{t=1}^T w_t\Delta y_t\\ \widetilde{\mu}_{2i} &= \mu_{2i}+y_{T_{2i-1}^0}-\mu_{2i-1},\ i=1,...,m/2.\end{aligned}$$

Then, $h_{t-1}=y_{t-1}-\widetilde{\mu}_{2i},\ t\in[T_{2i-1}^0+1,T_{2i}^0]$. We have

$$\begin{aligned}SSR_0^{(1)} &= \textstyle\sum_{t=1}^T(\Delta y_t-w_t'\breve{\Pi})^2=\sum_{i=0}^{m/2}\sum_{t=T_{2i}^0+1}^{T_{2i+1}^0}\{w_t'(\Pi-\breve{\Pi})+e_t\}^2\\ &\quad +\textstyle\sum_{i=1}^{m/2}\sum_{t=T_{2i-1}^0+1}^{T_{2i}^0}\{(\alpha_{2i}-1)h_{t-1}+w_t'(\Pi-\breve{\Pi})+e_t\}^2\\ &= \textstyle\sum_{t=1}^T e_t^2+\sum_{i=1}^{m/2}(\alpha_{2i}-1)^2(\sum_{t=T_{2i-1}^0+1}^{T_{2i}^0}h_{t-1}^2)\\ &\quad +2(\Pi-\breve{\Pi})'\textstyle\sum_{i=1}^{m/2}(\alpha_{2i}-1)(\sum_{t=T_{2i-1}^0+1}^{T_{2i}^0}h_{t-1}w_t)\\ &\quad +(\Pi-\breve{\Pi})'(W'W)(\Pi-\breve{\Pi})+2(\Pi-\breve{\Pi})'W'e+2\textstyle\sum_{i=1}^{m/2}(\alpha_{2i}-1)(\sum_{t=T_{2i-1}^0+1}^{T_{2i}^0}h_{t-1}e_t).\end{aligned}$$

Let $\bar{Z}^{(1)}=diag(\widetilde{Z}_1^{(1)},...,\widetilde{Z}_{m+1}^{(1)})$, where $\widetilde{Z}_i^{(1)}$ is the first column of $\widetilde{Z}_i=(h_{T_{2i-1}^0},...,h_{T_{2i}^0-1})$ and the $[(m+1)\times 1]$ vector $\gamma_1=(0,\alpha_2-1,0,\alpha_4-1,..,0)'$. Noting that $\breve{\Pi}-\Pi=(W'W)^{-1}W'e+(W'W)^{-1}\sum_{i=1}^{m/2}(\alpha_{2i}-1)\sum_{t=T_{2i-1}^0+1}^{T_{2i}^0}h_{t-1}w_t$,

$$SSR_0^{(1)}=\sum_{t=1}^T e_t^2+\gamma_1'\bar{Z}^{(1)\prime}M_W\bar{Z}^{(1)}\gamma_1-e'W(W'W)^{-1}W'e+2\gamma_1'\bar{Z}^{(1)\prime}e. \tag{A.25}$$

Using (i) $e'W(W'W)^{-1}W'e=a_T^2[a_T^{-2}T^{-1/2}e'W(a_T^{-2}T^{-1}W'W)^{-1}a_T^{-2}T^{-1/2}W'e]=a_T^2O_p(1)=O_p(a_T^2)$; (ii) $\gamma_1'\bar{Z}^{(1)\prime}e=O_p(a_T^2T^{1/2})$, we have from $(A.25)$

$$a_T^{-2}SSR_0^{(1)}=a_T^{-2}\sum_{t=1}^T e_t^2+a_T^{-2}\gamma_1'\bar{Z}^{(1)\prime}M_W\bar{Z}^{(1)}\gamma_1+O_p(T^{1/2}).$$

Next, we have (with $\hat{\Pi}$ denoting the estimate of $\Pi$ under the alternative),

$$SSR_{1a,m}^{(1)}=\sum_{i=0}^{m/2}\sum_{t=T_{2i}^0+1}^{T_{2i+1}^0}\{w_t'(\Pi-\hat{\Pi})+e_t\}^2+\sum_{i=1}^{m/2}\sum_{t=T_{2i-1}^0+1}^{T_{2i}^0}\left\{\begin{array}{c}(\alpha_{2i}-\hat{\alpha}_{2i})(y_{t-1}-\bar{y}_{2i,-1})\\ +(w_t-\bar{w}_{2i})'(\Pi-\hat{\Pi})+e_t\end{array}\right\}^2.$$

Then, noting that

$$\begin{aligned}\hat{\Pi}-\Pi &= (W'W)^{-1}[\textstyle\sum_{i=1}^{m/2}(\alpha_{2i}-\hat{\alpha}_{2i})\sum_{t=T_{2i-1}^0+1}^{T_{2i}^0} w_t(y_{t-1}-\bar{y}_{2i,-1})+W'e] \\ &= [O_p(a_T^{-2}T^{-1})][O_p(a_T^2T^{1/2})+O_p(a_T^2T^{1/2})]=O_p(T^{-1/2})\end{aligned}$$

we can show, after some simplifications,

$$a_T^{-2}SSR_{1a,m}^{(1)}=a_T^{-2}\textstyle\sum_{t=1}^T e_t^2+O_p(1). \tag{A.26}$$

Combining $(A.25)$ and $(A.26)$,

$$a_T^{-2}(SSR_0^{(1)}-SSR_{1a,m}^{(1)})=a_T^{-2}\gamma_1'\bar{Z}^{(1)\prime}M_W\bar{Z}^{(1)}\gamma_1+O_p(T^{1/2}). \tag{A.27}$$

Now, since regime $2i$ $(i=1,...,m/2)$ is $I(0)$, $a_T^{-2}\gamma_1'\bar{Z}^{(1)\prime}M_W\bar{Z}^{(1)}\gamma_1=a_T^{-2}O_p(a_T^2T)=O_p(T)$. Since this term is positive and dominant in $(A.27)$, $F_{1a}(\lambda^0,m)$ diverges to positive infinity at rate $T$. Entirely analogous arguments can be used to show the divergence of $G_1(m)$ at rate $T$. The details are omitted.▲

**Proof of Theorem 4:** To prove this result, it is sufficient to show that

$$\lim_{T\to\infty}P(\min_{1\le i\le l+1}\{p_i^*\}<\eta_{l+1})\le\eta \tag{A.28}$$

as the rest of the proof follows the same arguments as in the proof of Theorem 2 in Kejriwal (2019). First, note that

$$\begin{aligned}P(\min_{1\le i\le l+1}\{p_i^*\} &< \eta_{l+1})=1-P(\min_{1\le i\le l+1}\{p_i^*\}\ge\eta_{l+1}) \\ &= 1-\Pi_{i=1}^{l+1}\left[P(p_i^*\ge\eta_{l+1})\right] \\ &= 1-\Pi_{i=1}^{l+1}\left[1-P(\{p_{1,W_1}^{*,(i)}<\eta_{l+1}\}\cap\{p_{1,G_1}^{*,(i)}<\eta_{l+1}\})\right]\end{aligned} \tag{A.29}$$

where the second equality follows from the independence of the test statistics across segments and the third from the fact that $p_i^*=\max\{p_{1,W_1}^{*,(i)},p_{1,G_1}^{*,(i)}\}$. Next, from Theorem 3(a,b), it follows that under the null hypothesis of $l$ breaks, we have for any segment $i\in\{1,...,l+1\}$,

$$\begin{aligned}P(\{p_{1,W_1}^{*,(i)} &< \eta_{l+1}\}\cap\{p_{1,G_1}^{*,(i)}<\eta_{l+1}\})\le P(\{p_{1,W_1}^{*,(i)}<\eta_{l+1}\})\to\eta_{l+1} \text{ if } i \text{ is } I(1), \\ P(\{p_{1,W_1}^{*,(i)} &< \eta_{l+1}\}\cap\{p_{1,G_1}^{*,(i)}<\eta_{l+1}\})\le P(\{p_{1,G_1}^{*,(i)}<\eta_{l+1}\})\to\eta_{l+1} \text{ if } i \text{ is } I(0).\end{aligned} \tag{A.30}$$

Thus, using $(A.30)$ in $(A.29)$, we have $\lim_{T\to\infty}P(\min_{1\le i\le l+1}\{p_i^*\}<\eta_{l+1})\le 1-\left[1-\eta_{l+1}\right]^{l+1}=\eta$, which proves $(A.28)$.▲

**Proof of Theorem 5:** We prove the result for Model 2a and $k$ even. The proofs for the other tests follow using entirely analogous arguments. For $t\in[T_{2i-1}+1,T_{2i}]$, define

$$\begin{aligned}\breve{y}_t &= y_t-\bar{y}_{2i}-\frac{\sum_{t=T_{2i-1}+1}^{T_{2i}}(y_t-\bar{y}_{2i})(t-\bar{t}_{2i})}{\sum_{t=T_{2i-1}+1}^{T_{2i}}(t-\bar{t}_{2i})^2}(t-\bar{t}_{2i}) \\ \breve{y}_{t-1} &= y_{t-1}-\bar{y}_{2i,-1}-\frac{\sum_{t=T_{2i-1}+1}^{T_{2i}}(y_{t-1}-\bar{y}_{2i,-1})(t-\bar{t}_{2i})}{\sum_{t=T_{2i-1}+1}^{T_{2i}}(t-\bar{t}_{2i})^2}(t-\bar{t}_{2i})\end{aligned}$$

Following the proof of Theorem 1 and that of Theorem 1 in KPZ, it is easy to show that

$$\widetilde{SSR}_0^{(1)} - SSR_{2a,k}^{(1)} = -\left(T^{-1/2}\sum_{t=1}^{T} e_t\right)^2 + \sum_{i=0}^{k/2}\left[\frac{T}{T_{2i+1}-T_{2i}}(T^{-1/2}\sum_{t=T_{2i}+1}^{T_{2i+1}} e_t)^2\right] \quad (A.31)$$
$$+ \sum_{i=1}^{k/2}\left[\begin{array}{c} \frac{\{\sum_{t=T_{2i-1}+1}^{T_{2i}}\breve{y}_{t-1}e_t\}^2}{\sum_{t=T_{2i-1}+1}^{T_{2i}}\breve{y}_{t-1}^2} + \frac{T}{T_{2i}-T_{2i-1}}(T^{-1/2}\sum_{t=T_{2i-1}+1}^{T_{2i}} e_t)^2 \\ + \frac{\{\sum_{t=T_{2i-1}+1}^{T_{2i}}(t-\bar{t}_{2i})e_t\}^2}{\sum_{t=T_{2i-1}+1}^{T_{2i}}(t-\bar{t}_{2i})^2}\end{array}\right] + o_p(1)$$

It then follows that

$$F_{2a}(\lambda,k) = \frac{(T-2k-1)(\widetilde{SSR}_0^{(1)} - SSR_{2a,k}^{(1)})}{(2k)SSR_{2a,k}^{(1)}} \quad (A.32)$$
$$\xrightarrow{w} \frac{1}{2k}\left[\begin{array}{c} -B_{g,1}^2(1) + \sum_{i=0}^{k/2}\left[\frac{1}{\lambda_{2i+1}-\lambda_{2i}}\{B_{g,1}(\lambda_{2i+1}) - B_{g,1}(\lambda_{2i})\}^2\right] \\ \sum_{i=1}^{k/2}\left[\begin{array}{c} \frac{\{\int_{\lambda_{2i-1}}^{\lambda_{2i}}\breve{B}_{g,1}^{(2i)}(r)dB_{g,1}(r)\}^2}{\int_{\lambda_{2i-1}}^{\lambda_{2i}}\left[\breve{B}_{g,1}^{(2i)}(r)\right]^2 dr} + \frac{1}{\lambda_{2i}-\lambda_{2i-1}}\{B_{g,1}(\lambda_{2i}) - B_{g,1}(\lambda_{2i-1})\}^2 \\ + \frac{\left[\int_{\lambda_{2i-1}}^{\lambda_{2i}}\{r-(\lambda_{2i}-\lambda_{2i-1})^{-1}\int_{\lambda_{2i-1}}^{\lambda_{2i}} sds\}dB_{g,1}(r)\right]^2}{\int_{\lambda_{2i-1}}^{\lambda_{2i}}\{r-(\lambda_{2i}-\lambda_{2i-1})^{-1}\int_{\lambda_{2i-1}}^{\lambda_{2i}} sds\}^2 dr}\end{array}\right]\end{array}\right] \equiv F_{2a}^0(\lambda,k)$$

As for the non-trending case, the limiting distribution of the KPZ test is non-pivotal, depending on the unknown volatility process. We next prove that the bootstrap test statistic $F_{2a}^*(\lambda,k) = [(T-2k-1)(\widetilde{SSR}_0^{*,(1)} - SSR_{2a,k}^{*,(1)})]/[(2k)SSR_{2a,k}^{*,(1)}]$ follows the same distribution as $F_{2a}(\lambda,k)$. We have

$$\widetilde{SSR}_0^{*,(1)} - SSR_{2a,k}^{*,(1)} = -\left(T^{-1/2}\sum_{t=1}^{T} e_t^{(1)}\right)^2 + \sum_{i=0}^{k/2}\left[\frac{T}{T_{2i+1}-T_{2i}}(T^{-1/2}\sum_{t=T_{2i}+1}^{T_{2i+1}} e_t^{(1)})^2\right] \quad (A.33)$$
$$+ \sum_{i=1}^{k/2}\left[\begin{array}{c} \frac{\{\sum_{t=T_{2i-1}+1}^{T_{2i}}\breve{y}_{t-1}^{(1)}e_t^{(1)}\}^2}{\sum_{t=T_{2i-1}+1}^{T_{2i}}[\breve{y}_{t-1}^{(1)}]^2} + \frac{T}{T_{2i}-T_{2i-1}}(T^{-1/2}\sum_{t=T_{2i-1}+1}^{T_{2i}} e_t^{(1)})^2 \\ + \frac{\{\sum_{t=T_{2i-1}+1}^{T_{2i}}(t-\bar{t}_{2i})e_t^{(1)}\}^2}{\sum_{t=T_{2i-1}+1}^{T_{2i}}(t-\bar{t}_{2i})^2}\end{array}\right]$$

Similar to the analysis in the proof of Theorem 3(a), we can establish that

$$\begin{aligned}
&a_T^{-2}T^{-2}\sum_{t=T_{2i-1}+1}^{T_{2i}}[\breve{y}_{t-1}^{(1)}]^2 \xrightarrow{w}_p \widetilde{g}(1)^2\int_{\lambda_{2i-1}}^{\lambda_{2i}}[\breve{B}_{g,1}^{(2i)}(s)]^2 ds \\
&a_T^{-2}T^{-1}\sum_{t=T_{2i-1}+1}^{T_{2i}}\{\breve{y}_{t-1}^{(1)}e_t^{(1)}\} \xrightarrow{w}_p \widetilde{g}(1)^2\int_{\lambda_{2i-1}}^{\lambda_{2i}}\breve{B}_{g,1}^{(2i)}(s)dB_{g,1}(s) \\
&a_T^{-1}T^{-1/2}\sum_{t=T_{2i-1}+1}^{T_{2i}} e_t^{(1)} \xrightarrow{w}_p (\lambda_{2i}-\lambda_{2i-1})^{-1}\widetilde{g}(1)[B_{g,1}(\lambda_{2i}) - B_{g,1}(\lambda_{2i-1})] \\
&a_T^{-1}T^{-3/2}\sum_{t=T_{2i-1}+1}^{T_{2i}}(t-\bar{t}_{2i})e_t^{(1)} \xrightarrow{w}_p \widetilde{g}(1)\int_{\lambda_{2i-1}}^{\lambda_{2i}}\{r-(\lambda_{2i}-\lambda_{2i-1})^{-1}\int_{\lambda_{2i-1}}^{\lambda_{2i}} sds\}dB_{g,1}(r) \\
&T^{-3}\sum_{t=T_{2i-1}+1}^{T_{2i}}(t-\bar{t}_{2i})^2 \xrightarrow{w}_p \int_{\lambda_{2i-1}}^{\lambda_{2i}}\{r-(\lambda_{2i}-\lambda_{2i-1})^{-1}\int_{\lambda_{2i-1}}^{\lambda_{2i}} sds\}^2 dr \qquad (A.34)
\end{aligned}$$

Substituting $(A.34)$ in $(A.33)$ and noting that $a_T^{-2}(T-2k-1)^{-1}SSR_{2a,k}^{*,(1)} \xrightarrow{p} \widetilde{g}(1)^2$, we have $F_{2a}^*(\lambda,k) \xrightarrow{w}_p F_{2a}^0(\lambda,k)$. Next, for the $I(0)$ null $\widetilde{H}_0^{(0)}$, the test statistic is $G_2(\lambda,k) = [T-3(k+1)](\widetilde{SSR}_0^{(0)} - SSR_{2,k}^{(0)})/[kSSR_{2,k}^{(0)}]$. From the proof of Theorem 2 in Kejriwal (2019) and that of Theorem 2 above, after some algebra, it can be shown that

$$\begin{aligned}\widetilde{SSR}_0^{(0)} - SSR_{2,k}^{(0)} &= -(Y_{-1}'M_QY_{-1})^{-1}(e'M_QY_{-1})^2 - e'P_Qe \\ &\quad + \sum_{i=1}^{k+1}\left[(Y_{i,-1}'M_{Q_i}Y_{i,-1})^{-1}(e_i'M_{Q_i}Y_{i,-1})^2 + e_i'P_{Q_i}e_i\right] + o_p(1) \end{aligned} \quad (\text{A.35})$$

where, for $i=1,...,k+1$, $Q_i = (q_{T_{i-1}+1},...,q_{T_i})'$ with $q_t = (1,t)'$ for $t = T_{i-1}+1,...,T_i$, $Q = (q_1,...,q_T)'$. Denoting $J(r) = (1,r)'$ and using the following results:

$$\begin{aligned}
&a_T^{-2}T^{-1}Y_{-1}'M_QY_{-1} \xrightarrow{p} \widetilde{g}(1)^2\Omega_{11} \\
&a_T^{-2}T^{-1/2}Y_{-1}'M_Qe \xrightarrow{w} \widetilde{g}(1)^2\Omega_{11}^{1/2}B_{g,2}(1) \\
&a_T^{-2}T^{-1}Y_{i,-1}'M_{Q_i}Y_{i,-1} \xrightarrow{p} \widetilde{g}(\lambda_i)^2\Omega_{11} \\
&a_T^{-2}T^{-1/2}Y_{i,-1}'M_{Q_i}e_i \xrightarrow{w} \widetilde{g}(\lambda_i)^2\Omega_{11}^{1/2}[B_{g,2}(\lambda_i) - B_{g,2}(\lambda_{i-1})] \\
&a_T^{-2}e'P_Qe \xrightarrow{w} \widetilde{g}(1)^2\left(\int_0^1 J(r)dB_{g,1}(r)\right)'\left(\int_0^1 J(r)J(r)'dr\right)^{-1}\left(\int_0^1 J(r)dB_{g,1}(r)\right) \\
&a_T^{-2}\sum_{i=1}^{k+1} e_i'P_{Q_i}e_i \xrightarrow{w} \widetilde{g}(1)^2\sum_{i=1}^{k+1}\left(\int_{\lambda_{i-1}}^{\lambda_i} J(r)dB_{g,1}(r)\right)'\left(\int_{\lambda_{i-1}}^{\lambda_i} J(r)J(r)'dr\right)^{-1}\left(\int_{\lambda_{i-1}}^{\lambda_i} J(r)dB_{g,1}(r)\right) \\
&a_T^{-2}[T-3(k+1)]^{-1}SSR_{2,k}^{(0)} \xrightarrow{p} \widetilde{g}(1)^2
\end{aligned} \quad (\text{A.36})$$

we have

$$G_2(\lambda,k) \xrightarrow{w}$$

$$\frac{1}{k}\left[\begin{array}{c} \sum_{i=1}^{k} \frac{\left\{\widetilde{g}(\lambda_n)^2 B_{g,2}(\lambda_{n+1}) - \widetilde{g}(\lambda_{n+1})^2 B_{g,2}(\lambda_n)\right\}^2}{\widetilde{g}(\lambda_n)^2\widetilde{g}(\lambda_{n+1})^2\{\widetilde{g}(\lambda_{n+1})^2 - \widetilde{g}(\lambda_n)^2\}} \\ -\left(\int_0^1 J(r)dB_{g,1}(r)\right)'\left(\int_0^1 J(r)J(r)'dr\right)^{-1}\left(\int_0^1 J(r)dB_{g,1}(r)\right) \\ +\sum_{i=1}^{k+1}\left(\int_{\lambda_{i-1}}^{\lambda_i} J(r)dB_{g,1}(r)\right)'\left(\int_{\lambda_{i-1}}^{\lambda_i} J(r)J(r)'dr\right)^{-1}\left(\int_{\lambda_{i-1}}^{\lambda_i} J(r)dB_{g,1}(r)\right) \end{array}\right] \equiv G_2^0(\lambda,k)$$

Next, note that the bootstrap statistic $G_2^*(\lambda,k)$ is of the same form as $(A.35)$ with regressors [regressand] replaced by the bootstrap data $e_{-1}^{(0)}$ $[e^{(0)}]$. Similar to the convergence results in $(A.34)$, it is easy to establish that the same convergence results in $(A.36)$ hold for the bootstrap analogues. Thus, we have $G_2^*(\lambda,k) \xrightarrow{w}_p G_2^0(\lambda,k)$. The rest of the proof uses arguments entirely analogous to those used to prove Theorem 3(b)-(c). ▲

## Supplement B: Detailed Simulation Results

Supplement B presents detailed simulation results to assess the finite sample performance of our procedures and to provide a comparison with existing approaches. Following Cavaliere and Taylor (2008b, CT henceforth), we consider the following three specifications for the volatility process: Model 1 (Single Volatility Break): $\sigma_t = \sigma_0^* + (\sigma_1^* - \sigma_0^*)I(t \geq 0.5T)$; Model 2 (Trending Volatility): $\sigma_t = \sigma_0^* + (\sigma_1^* - \sigma_0^*)(t-1)/(T-1)$; Model 3 (Near-Integrated Stochastic Volatility): $\sigma_t = \sigma_0^* \exp(0.5\upsilon b_t/\sqrt{T})$, $b_t = (1-c/T)b_{t-1} + k_t$, $k_t \sim i.i.d.\ N(0,1)$, $b_0 = 0$. We set $\sigma_0^* = 1$ in all cases, $\delta := \sigma_0^*/\sigma_1^* \in \{1, 1/3, 3\}$ for Models 1 and 2, $\upsilon = 5$ and $c \in \{0, 10\}$ for Model 3. Next, we generate an ARMA(1,1) process $\{z_t\}_{t=1}^T$ as: $z_t = \rho z_{t-1} + e_t - \theta e_{t-1}$, $z_0 = 0$, $e_t = \sigma_t \varepsilon_t$ with $\varepsilon_t \sim i.i.d.\ N(0,1)$. While our theory does not formally allow for moving average processes, we nevertheless include this case in our simulations as a robustness check. The wild bootstrap is implemented using a two point distribution, i.e., $v_t \in \{-1, 1\}$ with equal probability. We also experimented with the standard normal distribution for $v_t$ but found that our tests perform noticeably better when using the two-point distribution relative to the normal. The level of trimming is set at $\epsilon = .15$, $T \in \{200, 400\}$ and 1000 replications are used. We report results for the non-trending case only (those for the trending case are qualitatively similar). The lag length in the KPZ and BP procedures is selected using BIC with maximal value set to five.[1] We report the performance of the tests $H^*(k,\eta)$; $k = 1, 2$ and $Hmax_1^*(\eta) = \max\{H^*(1,\eta), H^*(2,\eta)\}$ as well as their non-robust (homoskedasticity-based) asymptotic counterparts $H(k,\eta)$; $k = 1, 2$ and $Hmax_1(\eta) = \max\{H(1,\eta), H(2,\eta)\}$. The ratio-based bootstrap tests of CT are designed to test the $I(0)$ null hypothesis while our tests allow the process to be either $I(1)$ or $I(0)$ under the null. Further, while our tests are based on a finite order autoregressive model, the CT tests are non-parametric and based on a mixing-type assumption for the innovations. Given that conducting a full and fair comparison of tests with different underlying models and null hypotheses is not possible, we did not include the CT tests in our analysis.

**Finite Sample Size.** With no persistence change, $\{y_t\}$ is generated by DGP-0: $y_t = \alpha y_{t-1} + z_t$, $y_0 = 0$. Table B-1 reports the empirical size of 5% asymptotic and bootstrap tests. The asymptotic tests are considerably oversized indicating their lack of robustness to nonstationary volatility, consistent with the large sample results in Section 4. In contrast, the proposed bootstrap tests are robust to $I(1)$ or $I(0)$ processes maintaining empirical size close to the nominal 5% level across the different volatility specifications. The same is generally true for the different error structures considered. The $H^*$ tests are more accurately sized when $\alpha \in \{.5, 1\}$ but less so when $\alpha = .7$, since the tests are a hybrid of the KPZ and BP tests which each have size close to 5% when $\alpha = 1$ and $\alpha = .5$, respectively. When

[1] We also experimented with larger values and found that they yielded comparable size but lower power, especially for the multiple break and sequential tests. The BIC was computed under the null model for each test. No qualitative differences were observed if the lag selection was implemented under the alternative model. The modified information proposed by Cavaliere et al. (2015) yielded no improvement in our setup.

$\alpha = .7$, the BP tests are mildly over-sized while the KPZ tests diverge at rate $T$, hence the mild size distortions. Similar reasoning explains the slightly higher sizes for the hybrid tests when $T$ increases, especially when $\alpha = .7$ and with MA(1) errors.

**Finite Sample Power.** We consider DGPs with one and two breaks. The results are reported only for the case $\rho = \theta = 0$ and are briefly summarized for the other cases. The DGPs in the one break case are DGP-1: $y_t = \alpha y_{t-1} + z_t$ if $t \leq \lfloor T\lambda_1^0 \rfloor$, $y_t = y_{t-1} + z_t$ otherwise; DGP-2: $y_t = y_{t-1} + z_t$ if $t \leq \lfloor T\lambda_1^0 \rfloor$, $y_t - y_{\lfloor T\lambda_1^0 \rfloor} = \alpha(y_{t-1} - y_{\lfloor T\lambda_1^0 \rfloor}) + z_t$, otherwise; DGP-3: $y_t = \alpha_1 y_{t-1} + z_t$ if $t \leq \lfloor T\lambda_1^0 \rfloor$, $y_t = \alpha_2 y_{t-1} + z_t$, otherwise. For DGP-1 and DGP-2, $\alpha \in \{.5, .7\}$ and for DGP-3, $\alpha_1, \alpha_2 \in \{.2, .9\}$. We define $\alpha = \alpha_2 - \alpha_1$. The break fraction is $\lambda_1^0 = .5$. Table B-2 reports the size-adjusted power of the tests. Several features are worth noting. First, the bootstrap tests are broadly comparable to their asymptotic counterparts, with neither class of tests uniformly dominating the other. Second, the proposed tests are generally powerful against the different persistence change alternatives; an exception being the case of an $I(1)$ regime with high volatility (e.g., DGP-1 with $\delta = 1/3$ and DGP-2 with $\delta = 3$). This occurs since the process is dominated by the $I(1)$ regime and the tests behave as with a stable $I(1)$ process. In Table B-3, we show that power improves considerably if the $I(0)$ regime is longer and/or the volatility shift is less prominent. Third, the proposed tests have substantial power against $I(0)$-preserving breaks (DGP-3), a feature that distinguishes these tests from most existing persistence change tests (e.g., the ratio-based tests) that only have trivial power against such breaks sinch they are designed to detect changes between $I(1)$ and $I(0)$ regimes. Hence, using the KPZ tests to control size in the $I(1)$ case causes little power loss relative to using the BP tests in isolation. Fourth, the hybrid tests are generally more powerful with deterministic (Models 1 and 2) rather than stochastic volatility (Model 3). With serially correlated errors, the results (Tables B-4 and B-5) are qualitatively similar except that power is lower for all the tests relative to $\rho = \theta = 0$. With two breaks, the DGPs are specified as follows:

| | For $t \leq \lfloor T\lambda_1^0 \rfloor$ | For $\lfloor T\lambda_1^0 \rfloor + 1 \leq t \leq \lfloor T\lambda_2^0 \rfloor$ | For $t \geq \lfloor T\lambda_2^0 \rfloor + 1$ |
|---|---|---|---|
| DGP-4 | $y_t = y_{t-1} + z_t$ | $y_t - y_{\lfloor T\lambda_1^0 \rfloor} = \alpha(y_{t-1} - y_{\lfloor T\lambda_1^0 \rfloor}) + z_t$ | $y_t = y_{t-1} + z_t$ |
| DGP-5 | $y_t = \alpha y_{t-1} + z_t$ | $y_t = y_{t-1} + z_t$ | $y_t - y_{\lfloor T\lambda_2^0 \rfloor} = \alpha(y_{t-1} - y_{\lfloor T\lambda_2^0 \rfloor}) + z_t$ |
| DGP-6 | $y_t = \alpha_1 y_{t-1} + z_t$ | $y_t = \alpha_2 y_{t-1} + z_t$ | $y_t = \alpha_1 y_{t-1} + z_t$ |

The true break fractions are $(\lambda_1^0, \lambda_2^0) = (.3, .8)$. The results reported in Table B-6 are broadly similar to the one break case. The effect of underspecifying the number of breaks can be seen by comparing the power of $H^*(1)$ and $H^*(2)$ for DGPs with two breaks, where the former is generally less powerful than the latter, though not in all cases. Interestingly, when the first regime has lower persistence relative to the second, $H^*(1)$ has higher power than $H^*(2)$ even though the former is based on a misspecified model. However, $Hmax_1^*$ has adequate power in most cases, often close to that of the most powerful test amongst $H^*(1)$ and

$H^*(2)$. This highlights the practical advantage of using $Hmax_1^*$ to detect the presence of at least one break. Tables B-7 and B-8 provide results with serial correlation showing the results to be qualitatively similar.

**Number of Breaks.** We assess the adequacy of the sequential algorithm to estimate the number of breaks with data generated by DGP 0-6. We set $A = 2$ and $\eta = .10$. The results are in Table B-9, with $P_c$ and $P_o$ denoting the probability of correct selection and over-estimation, respectively. The procedure is generally reliable in the stable case (DGP-0) or with a single break (DGP 1-3). Its performance deteriorates in the two breaks case when the probability of underestimation can be non-negligible. For instance, in DGP-5 with an abrupt increase in volatility (Model 1, $\delta = 1/3$), the breakpoint estimate used to partition the sample is typically close to the second true breakpoint, so that the first segment includes an $I(0)$ to $I(1)$ break while the second is $I(0)$. Whether a second break is selected depends on the power of the single break test in the first segment, which is relatively low (Table B-2). Similarly, with decreasing volatility, the breakpoint is estimated near the first true date so that selecting an additional break depends on the power of the single break test in the $I(1)$-$I(0)$ case. Results reported in Table B-10 show a notable improvement as the magnitude of the volatility shift decreases and/or the volatility shift occurs near the second persistence break in the increasing volatility case and near the first break otherwise. Further refinement of the algorithm is a potentially interesting topic for future research.

**Disentangling Trend and Persistence Shifts.** Table B-11 reports the probabilities of selecting the true model based on the procedure proposed in Section 6 for disentangling mean shifts and persistence breaks. In addition to DGPs 1-3, a DGP with a pure mean shift, denoted DGP-$0_\mu$, is considered where the data are generated as: $y_t = (\Delta\mu)I(t > \lfloor T\lambda_1^0 \rfloor) + \alpha y_{t-1} + e_t$, where $e_t = \sigma_t \varepsilon_t$, $\varepsilon_t \sim i.i.d.\ N(0,1)$ and the same three specifications for $\sigma_t$ (Models 1-3) are used. The findings indicate that performance is generally satisfactory and improves as $T$ increases.

*Notes to Tables*

1. Table B-1 reports the empirical size of asymptotic and bootstrap tests with nominal size 5%. The tests $H_1, H_2, H_{max}$ are the tests of Kejriwal et al. (2013) and the tests $H_1^*, H_2^*, H_{max}^*$ are their bootstrap counterparts.

2. Table B-2 reports the size-adjusted power of asymptotic and bootstrap tests with nominal size 5% in the single break case with breakpoint $\lambda_1^0 = .5$ and serially uncorrelated errors ($\rho = \theta = 0$).

3. Table B-3 reports the size-adjusted power of 5% bootstrap tests under different change points and volatility intensity in the single break case with breakpoint $\lambda_1^0 = .5$ and serially uncorrelated errors ($\rho = \theta = 0$) with abrupt volatility change (Model 1).

4. Table B-4 reports the size-adjusted power of asymptotic and bootstrap tests with nominal size 5% in the single break case with breakpoint $\lambda_1^0 = .5$ and AR(1) errors ($\rho = .5, \theta = 0$).

5. Table B-5 reports the size-adjusted power of asymptotic and bootstrap tests with nominal size 5% in the single break case with breakpoint $\lambda_1^0 = .5$ and MA(1) errors ($\rho = 0, \theta = .5$).

6. Table B-6 reports the size-adjusted power of asymptotic and bootstrap tests with nominal size 5% in the two breaks case with breakpoint vector $(\lambda_1^0, \lambda_2^0) = (.3, .8)$ and serially uncorrelated errors ($\rho = \theta = 0$).

7. Table B-7 reports the size-adjusted power of asymptotic and bootstrap tests with nominal size 5% in the two breaks case with breakpoint vector $(\lambda_1^0, \lambda_2^0) = (.3, .8)$ and AR(1) errors ($\rho = .5, \theta = 0$).

8. Table B-8 reports the size-adjusted power of asymptotic and bootstrap tests with nominal size 5% in the two breaks case with breakpoint vector $(\lambda_1^0, \lambda_2^0) = (.3, .8)$ and MA(1) errors ($\rho = 0, \theta = .5$).

9. Table B-9 reports the probabilities of selecting the true number of breaks from the sequential procedure with serially uncorrelated errors ($\rho = \theta = 0$) and level $\eta = .10$.

10. Table B-10 reports the probabilities of selecting the true number of breaks from the sequential procedure under different abrupt volatility break points and intensities in the two breaks case with breakpoint vector $(\lambda_1^0, \lambda_2^0) = (.3, .8)$, serially uncorrelated errors ($\rho = \theta = 0$) and level $\eta = .10$.

11. Table B-11 reports the probabilities of selecting the true model based on the procedure proposed in Section 6 for disentangling mean shifts and persistence breaks.

Table B-1: Empirical size of asymptotic and bootstrap tests, [$m = 0$, 5% nominal size]

| $T$ | $(\rho,\theta)$ | Test | $\alpha = 1$ | | | | | | | $\alpha = 0.5$ | | | | | | | $\alpha = 0.7$ | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | Model 1: $\delta$ | | | Model 2: $\delta$ | | Model 3: $c$ | | Model 1: $\delta$ | | | Model 2: $\delta$ | | Model 3: $c$ | | Model 1: $\delta$ | | | Model 2: $\delta$ | | Model 3: $c$ | |
| | | $\delta/c$ | 1 | 1/3 | 3 | 1/3 | 3 | 0 | 10 | 1 | 1/3 | 3 | 1/3 | 3 | 0 | 10 | 1 | 1/3 | 3 | 1/3 | 3 | 0 | 10 |
| 200 | (0,0) | $H_1$ | .06 | .39 | .21 | .26 | .11 | .49 | .32 | .05 | .15 | .14 | .10 | .08 | .28 | .15 | .07 | .18 | .15 | .14 | .10 | .31 | .19 |
| | | $H_2$ | .04 | .33 | .35 | .07 | .14 | .41 | .38 | .03 | .15 | .14 | .07 | .05 | .23 | .15 | .05 | .21 | .19 | .09 | .07 | .25 | .20 |
| | | $H_{max}$ | .05 | .44 | .37 | .17 | .16 | .56 | .46 | .04 | .19 | .16 | .10 | .08 | .31 | .17 | .06 | .24 | .19 | .14 | .09 | .34 | .22 |
| | | $H_1^*$ | .06 | .06 | .04 | .06 | .03 | .06 | .06 | .05 | .03 | .04 | .04 | .04 | .03 | .03 | .08 | .06 | .04 | .07 | .05 | .04 | .03 |
| | | $H_2^*$ | .06 | .05 | .06 | .06 | .06 | .05 | .06 | .04 | .02 | .01 | .05 | .03 | .03 | .02 | .07 | .03 | .03 | .06 | .04 | .03 | .04 |
| | | $H_{max}^*$ | .06 | .05 | .05 | .06 | .05 | .05 | .05 | .05 | .03 | .02 | .04 | .03 | .03 | .03 | .08 | .05 | .03 | .06 | .04 | .04 | .03 |
| | (0.5,0) | $H_1$ | .06 | .42 | .24 | .27 | .12 | .51 | .33 | .07 | .18 | .16 | .13 | .11 | .31 | .17 | .09 | .22 | .18 | .16 | .13 | .31 | .19 |
| | | $H_2$ | .05 | .33 | .36 | .07 | .16 | .44 | .35 | .03 | .19 | .19 | .05 | .07 | .23 | .18 | .05 | .24 | .22 | .09 | .09 | .26 | .21 |
| | | $H_{max}$ | .06 | .46 | .39 | .17 | .18 | .58 | .45 | .05 | .23 | .20 | .11 | .10 | .33 | .20 | .08 | .29 | .24 | .14 | .12 | .36 | .24 |
| | | $H_1^*$ | .06 | .06 | .05 | .06 | .05 | .05 | .06 | .08 | .04 | .04 | .05 | .05 | .04 | .04 | .10 | .05 | .05 | .06 | .06 | .04 | .04 |
| | | $H_2^*$ | .08 | .07 | .07 | .05 | .07 | .06 | .06 | .04 | .02 | .03 | .04 | .03 | .03 | .03 | .07 | .04 | .04 | .07 | .05 | .04 | .03 |
| | | $H_{max}^*$ | .07 | .07 | .07 | .06 | .07 | .06 | .06 | .07 | .04 | .02 | .04 | .04 | .03 | .03 | .09 | .04 | .04 | .06 | .04 | .04 | .04 |
| | (0,0.5) | $H_1$ | .05 | .35 | .23 | .21 | .12 | .48 | .32 | .03 | .12 | .14 | .08 | .07 | .27 | .13 | .09 | .20 | .18 | .16 | .12 | .31 | .19 |
| | | $H_2$ | .05 | .26 | .29 | .06 | .10 | .37 | .31 | .01 | .07 | .08 | .02 | .02 | .17 | .12 | .04 | .16 | .15 | .07 | .07 | .23 | .15 |
| | | $H_{max}$ | .06 | .39 | .32 | .14 | .12 | .53 | .39 | .03 | .13 | .14 | .07 | .07 | .27 | .15 | .08 | .24 | .23 | .14 | .11 | .33 | .21 |
| | | $H_1^*$ | .08 | .06 | .05 | .06 | .04 | .05 | .06 | .04 | .02 | .03 | .02 | .03 | .02 | .02 | .10 | .04 | .06 | .05 | .05 | .04 | .05 |
| | | $H_2^*$ | .06 | .05 | .04 | .06 | .05 | .06 | .04 | .01 | .01 | .01 | .02 | .01 | .02 | .01 | .06 | .02 | .02 | .05 | .03 | .02 | .03 |
| | | $H_{max}^*$ | .06 | .06 | .04 | .06 | .04 | .05 | .05 | .03 | .01 | .02 | .02 | .02 | .02 | .02 | .09 | .03 | .03 | .05 | .04 | .02 | .03 |
| 400 | (0,0) | $H_1$ | .05 | .40 | .24 | .25 | .13 | .53 | .35 | .06 | .18 | .17 | .14 | .10 | .32 | .22 | .07 | .20 | .18 | .14 | .12 | .35 | .22 |
| | | $H_2$ | .05 | .38 | .44 | .09 | .18 | .45 | .42 | .06 | .21 | .20 | .09 | .08 | .30 | .25 | .06 | .24 | .25 | .11 | .10 | .32 | .26 |
| | | $H_{max}$ | .05 | .50 | .45 | .19 | .21 | .61 | .51 | .06 | .22 | .22 | .14 | .11 | .36 | .24 | .07 | .25 | .24 | .15 | .13 | .40 | .27 |
| | | $H_1^*$ | .04 | .05 | .06 | .04 | .04 | .05 | .05 | .07 | .05 | .05 | .06 | .05 | .04 | .06 | .08 | .06 | .05 | .07 | .05 | .05 | .06 |
| | | $H_2^*$ | .05 | .05 | .06 | .05 | .05 | .05 | .05 | .07 | .03 | .03 | .05 | .04 | .03 | .04 | .08 | .04 | .03 | .07 | .04 | .04 | .05 |
| | | $H_{max}^*$ | .04 | .04 | .05 | .04 | .05 | .05 | .05 | .06 | .05 | .05 | .05 | .04 | .04 | .04 | .08 | .06 | .04 | .08 | .05 | .04 | .06 |
| | (0.5,0) | $H_1$ | .05 | .41 | .25 | .26 | .13 | .53 | .36 | .07 | .22 | .20 | .16 | .13 | .34 | .22 | .09 | .22 | .22 | .18 | .15 | .37 | .23 |
| | | $H_2$ | .06 | .38 | .42 | .09 | .17 | .45 | .41 | .07 | .25 | .27 | .13 | .12 | .33 | .27 | .09 | .30 | .30 | .14 | .13 | .35 | .30 |
| | | $H_{max}$ | .06 | .49 | .45 | .18 | .19 | .61 | .50 | .08 | .26 | .26 | .17 | .15 | .39 | .27 | .10 | .28 | .29 | .20 | .18 | .42 | .30 |
| | | $H_1^*$ | .05 | .04 | .05 | .04 | .05 | .04 | .06 | .07 | .06 | .07 | .08 | .07 | .05 | .07 | .10 | .08 | .07 | .09 | .08 | .05 | .07 |
| | | $H_2^*$ | .05 | .06 | .05 | .06 | .05 | .05 | .05 | .09 | .04 | .04 | .09 | .05 | .05 | .05 | .11 | .06 | .05 | .10 | .06 | .04 | .06 |
| | | $H_{max}^*$ | .05 | .05 | .05 | .05 | .05 | .04 | .05 | .08 | .06 | .06 | .08 | .07 | .04 | .06 | .10 | .08 | .06 | .09 | .07 | .04 | .06 |
| | (0,0.5) | $H_1$ | .06 | .37 | .25 | .23 | .14 | .53 | .34 | .04 | .17 | .15 | .11 | .09 | .32 | .20 | .14 | .28 | .26 | .21 | .17 | .38 | .29 |
| | | $H_2$ | .06 | .37 | .37 | .09 | .15 | .44 | .38 | .04 | .18 | .16 | .08 | .07 | .29 | .22 | .14 | .33 | .30 | .18 | .17 | .35 | .32 |
| | | $H_{max}$ | .06 | .48 | .40 | .18 | .17 | .60 | .47 | .05 | .19 | .18 | .12 | .11 | .35 | .23 | .16 | .36 | .34 | .23 | .21 | .42 | .35 |
| | | $H_1^*$ | .06 | .05 | .07 | .04 | .07 | .05 | .06 | .05 | .06 | .04 | .05 | .04 | .04 | .04 | .15 | .10 | .09 | .10 | .10 | .07 | .08 |
| | | $H_2^*$ | .06 | .05 | .05 | .05 | .05 | .05 | .04 | .05 | .02 | .02 | .04 | .03 | .03 | .02 | .15 | .05 | .05 | .12 | .08 | .05 | .06 |
| | | $H_{max}^*$ | .06 | .05 | .05 | .04 | .05 | .04 | .04 | .05 | .05 | .04 | .05 | .03 | .04 | .03 | .16 | .09 | .09 | .11 | .10 | .05 | .07 |

Table B-2: Size-adjusted power [$m = 1$, $\rho = \theta = 0$, $\lambda_1^0 = 0.5$, 5%]

| $T$ | DGP | Test | Model 1: $\delta$ | | | Model 2: $\delta$ | | Model 3: $c$ | | Model 1: $\delta$ | | | Model 2: $\delta$ | | Model 3: $c$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\delta/c$ | 1 | 1/3 | 3 | 1/3 | 3 | 0 | 10 | 1 | 1/3 | 3 | 1/3 | 3 | 0 | 10 |
| | | | $\alpha = 0.5$ | | | | | | | $\alpha = 0.7$ | | | | | | |
| 200 | 1 | $H_1$ | .97 | .03 | .84 | .16 | .96 | .08 | .35 | .61 | .04 | .51 | .04 | .68 | .04 | .08 |
| | | $H_2$ | .60 | .03 | .17 | .15 | .36 | .04 | .08 | .16 | .05 | .12 | .05 | .14 | .03 | .04 |
| | | $H_{max}$ | .93 | .03 | .63 | .11 | .89 | .04 | .15 | .41 | .04 | .27 | .03 | .47 | .03 | .05 |
| | | $H_1^*$ | .97 | .04 | .84 | .28 | .95 | .26 | .53 | .61 | .04 | .46 | .06 | .65 | .10 | .22 |
| | | $H_2^*$ | .57 | .05 | .21 | .25 | .45 | .10 | .22 | .16 | .05 | .12 | .07 | .18 | .06 | .08 |
| | | $H_{max}^*$ | .91 | .05 | .67 | .23 | .89 | .19 | .38 | .43 | .04 | .30 | .05 | .48 | .08 | .13 |
| | 2 | $H_1$ | .99 | .77 | .22 | .96 | .69 | .14 | .56 | .85 | .45 | .15 | .70 | .29 | .07 | .25 |
| | | $H_2$ | .69 | .29 | .03 | .74 | .12 | .07 | .12 | .28 | .17 | .04 | .38 | .06 | .05 | .06 |
| | | $H_{max}$ | .98 | .68 | .06 | .94 | .37 | .08 | .28 | .73 | .37 | .05 | .66 | .11 | .05 | .10 |
| | | $H_1^*$ | 1.0 | .77 | .30 | .94 | .76 | .32 | .71 | .85 | .41 | .21 | .62 | .39 | .18 | .42 |
| | | $H_2^*$ | .71 | .35 | .06 | .71 | .23 | .16 | .31 | .30 | .18 | .05 | .36 | .10 | .09 | .16 |
| | | $H_{max}^*$ | .98 | .69 | .10 | .93 | .53 | .24 | .58 | .74 | .35 | .07 | .57 | .17 | .13 | .29 |
| | | | $\alpha_1 = 0.2, \alpha_2 = 0.9, \alpha = 0.7$ | | | | | | | $\alpha_1 = 0.9, \alpha_2 = 0.2, \alpha = -0.7$ | | | | | | |
| | 3 | $H_1$ | 1.0 | .05 | .86 | .69 | .99 | .18 | .65 | 1.0 | .83 | .33 | .99 | .93 | .24 | .75 |
| | | $H_2$ | .93 | .03 | .19 | .59 | .67 | .06 | .15 | .94 | .30 | .05 | .90 | .45 | .08 | .20 |
| | | $H_{max}$ | .99 | .04 | .76 | .62 | .98 | .12 | .41 | 1.0 | .79 | .11 | .98 | .81 | .16 | .53 |
| | | $H_1^*$ | 1.0 | .06 | .76 | .69 | .96 | .31 | .71 | 1.0 | .78 | .33 | .95 | .89 | .35 | .77 |
| | | $H_2^*$ | .88 | .04 | .23 | .60 | .71 | .15 | .41 | .93 | .32 | .06 | .82 | .59 | .18 | .46 |
| | | $H_{max}^*$ | .99 | .03 | .70 | .67 | .95 | .24 | .61 | 1.0 | .72 | .17 | .95 | .83 | .28 | .69 |
| | | | $\alpha = 0.5$ | | | | | | | $\alpha = 0.7$ | | | | | | |
| 400 | 1 | $H_1$ | 1.0 | .07 | 1.0 | .94 | 1.0 | .41 | .84 | 1.0 | .07 | .96 | .24 | .99 | .13 | .49 |
| | | $H_2$ | 1.0 | .03 | .65 | .75 | .95 | .12 | .26 | .75 | .04 | .44 | .16 | .64 | .07 | .12 |
| | | $H_{max}$ | 1.0 | .03 | 1.0 | .92 | 1.0 | .27 | .63 | .99 | .05 | .88 | .20 | .97 | .08 | .24 |
| | | $H_1^*$ | 1.0 | .07 | 1.0 | .95 | 1.0 | .51 | .84 | 1.0 | .05 | .95 | .33 | .99 | .31 | .62 |
| | | $H_2^*$ | .99 | .05 | .69 | .78 | .96 | .32 | .57 | .69 | .04 | .45 | .23 | .67 | .15 | .28 |
| | | $H_{max}^*$ | 1.0 | .06 | 1.0 | .93 | 1.0 | .42 | .75 | .99 | .05 | .87 | .26 | .98 | .22 | .43 |
| | 2 | $H_1$ | 1.0 | 1.0 | .44 | 1.0 | .99 | .45 | .91 | 1.0 | .94 | .21 | 1.0 | .74 | .25 | .70 |
| | | $H_2$ | 1.0 | .74 | .05 | .99 | .49 | .14 | .34 | .84 | .57 | .04 | .93 | .14 | .10 | .17 |
| | | $H_{max}$ | 1.0 | 1.0 | .13 | 1.0 | .97 | .32 | .79 | 1.0 | .90 | .06 | .99 | .45 | .17 | .40 |
| | | $H_1^*$ | 1.0 | 1.0 | .53 | 1.0 | .99 | .56 | .92 | 1.0 | .93 | .31 | .99 | .78 | .40 | .78 |
| | | $H_2^*$ | 1.0 | .78 | .06 | .99 | .67 | .39 | .67 | .83 | .59 | .05 | .91 | .24 | .23 | .41 |
| | | $H_{max}^*$ | 1.0 | 1.0 | .20 | 1.0 | .96 | .48 | .84 | 1.0 | .90 | .11 | .99 | .54 | .29 | .64 |
| | | | $\alpha_1 = 0.2, \alpha_2 = 0.9, \alpha = 0.7$ | | | | | | | $\alpha_1 = 0.9, \alpha_2 = 0.2, \alpha = -0.7$ | | | | | | |
| | 3 | $H_1$ | 1.0 | .39 | 1.0 | 1.0 | 1.0 | .45 | .95 | 1.0 | 1.0 | .83 | 1.0 | 1.0 | .50 | .97 |
| | | $H_2$ | 1.0 | .09 | .91 | .99 | 1.0 | .20 | .62 | 1.0 | .95 | .15 | 1.0 | .98 | .24 | .69 |
| | | $H_{max}$ | 1.0 | .15 | 1.0 | 1.0 | 1.0 | .39 | .87 | 1.0 | 1.0 | .42 | 1.0 | 1.0 | .43 | .94 |
| | | $H_1^*$ | 1.0 | .45 | 1.0 | 1.0 | 1.0 | .54 | .93 | 1.0 | 1.0 | .83 | 1.0 | 1.0 | .56 | .95 |
| | | $H_2^*$ | 1.0 | .17 | .91 | .99 | 1.0 | .42 | .82 | 1.0 | .92 | .24 | 1.0 | .98 | .44 | .84 |
| | | $H_{max}^*$ | 1.0 | .27 | 1.0 | 1.0 | 1.0 | .49 | .90 | 1.0 | 1.0 | .57 | 1.0 | 1.0 | .52 | .92 |

Table B-3: Size-adjusted power of bootstrap tests under different change points and volatility intensity, [Model 1, DGP 1, $m = 1$, $\rho = \theta = 0$, $\alpha = 0.5$]

| T | $\lambda_1^0$ | Test | Model 1: $\delta$ | | | | |
|---|---|---|---|---|---|---|---|
| | | | 1/5 | 1/3 | 1/2.5 | 1/1.5 | 1/1.1 |
| 200 | 0.2 | $H_1^*$ | .05 | .05 | .04 | .07 | .28 |
| | | $H_2^*$ | .04 | .04 | .04 | .05 | .09 |
| | | $H_{max}^*$ | .05 | .04 | .05 | .06 | .20 |
| | 0.3 | $H_1^*$ | .05 | .04 | .05 | .15 | .61 |
| | | $H_2^*$ | .04 | .04 | .04 | .08 | .20 |
| | | $H_{max}^*$ | .04 | .05 | .04 | .11 | .46 |
| | 0.5 | $H_1^*$ | .04 | .05 | .06 | .56 | .94 |
| | | $H_2^*$ | .04 | .04 | .04 | .24 | .55 |
| | | $H_{max}^*$ | .05 | .04 | .05 | .44 | .89 |
| | 0.6 | $H_1^*$ | .08 | .17 | .27 | .87 | .98 |
| | | $H_2^*$ | .03 | .04 | .07 | .41 | .70 |
| | | $H_{max}^*$ | .04 | .08 | .12 | .77 | .95 |
| | 0.7 | $H_1^*$ | .33 | .51 | .61 | .92 | .96 |
| | | $H_2^*$ | .19 | .24 | .27 | .58 | .77 |
| | | $H_{max}^*$ | .23 | .32 | .44 | .86 | .94 |
| | 0.9 | $H_1^*$ | .31 | .39 | .44 | .58 | .62 |
| | | $H_2^*$ | .30 | .31 | .32 | .43 | .47 |
| | | $H_{max}^*$ | .29 | .33 | .37 | .55 | .60 |
| 400 | 0.2 | $H_1^*$ | .06 | .06 | .06 | .30 | .88 |
| | | $H_2^*$ | .05 | .05 | .05 | .08 | .38 |
| | | $H_{max}^*$ | .05 | .05 | .05 | .16 | .74 |
| | 0.3 | $H_1^*$ | .06 | .06 | .06 | .73 | 1.0 |
| | | $H_2^*$ | .05 | .05 | .05 | .26 | .73 |
| | | $H_{max}^*$ | .05 | .05 | .05 | .55 | .98 |
| | 0.5 | $H_1^*$ | .06 | .08 | .20 | .99 | 1.0 |
| | | $H_2^*$ | .05 | .05 | .06 | .78 | .99 |
| | | $H_{max}^*$ | .05 | .05 | .10 | .97 | 1.0 |
| | 0.6 | $H_1^*$ | .58 | .84 | .94 | 1.0 | 1.0 |
| | | $H_2^*$ | .07 | .18 | .31 | .93 | 1.0 |
| | | $H_{max}^*$ | .30 | .65 | .83 | 1.0 | 1.0 |
| | 0.7 | $H_1^*$ | .96 | .99 | 1.0 | 1.0 | 1.0 |
| | | $H_2^*$ | .67 | .75 | .82 | .98 | 1.0 |
| | | $H_{max}^*$ | .88 | .96 | .98 | 1.0 | 1.0 |
| | 0.9 | $H_1^*$ | .84 | .88 | .90 | .93 | .93 |
| | | $H_2^*$ | .78 | .79 | .80 | .84 | .86 |
| | | $H_{max}^*$ | .80 | .84 | .87 | .93 | .93 |

Table B-4: Size-adjusted power [$m = 1$, $\rho = 0.5$, $\theta = 0$, $\lambda_1^0 = 0.5$]

| $T$ | DGP | Test | Model 1: $\delta$ | | | Model 2: $\delta$ | | Model 3: $c$ | | Model 1: $\delta$ | | | Model 2: $\delta$ | | Model 3: $c$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\delta/c$ | 1 | 1/3 | 3 | 1/3 | 3 | 0 | 10 | 1 | 1/3 | 3 | 1/3 | 3 | 0 | 10 |
| | | | $\alpha = 0.5$ | | | | | | | $\alpha = 0.7$ | | | | | | |
| 200 | 1 | $H_1$ | .82 | .06 | .56 | .09 | .80 | .06 | .20 | .39 | .06 | .33 | .06 | .48 | .04 | .07 |
| | | $H_2$ | .35 | .05 | .11 | .10 | .21 | .05 | .07 | .11 | .05 | .08 | .05 | .11 | .03 | .04 |
| | | $H_{max}$ | .67 | .06 | .29 | .06 | .62 | .05 | .09 | .25 | .05 | .15 | .04 | .29 | .03 | .05 |
| | | $H_1^*$ | .82 | .06 | .53 | .15 | .75 | .16 | .32 | .39 | .05 | .25 | .06 | .43 | .06 | .13 |
| | | $H_2^*$ | .30 | .05 | .10 | .17 | .26 | .10 | .16 | .10 | .05 | .07 | .08 | .11 | .06 | .08 |
| | | $H_{max}^*$ | .63 | .06 | .29 | .13 | .59 | .12 | .22 | .22 | .05 | .12 | .07 | .26 | .05 | .09 |
| | 2 | $H_1$ | .95 | .40 | .22 | .81 | .50 | .10 | .38 | .74 | .26 | .17 | .51 | .26 | .06 | .21 |
| | | $H_2$ | .45 | .16 | .04 | .49 | .10 | .07 | .10 | .21 | .11 | .05 | .28 | .05 | .05 | .06 |
| | | $H_{max}$ | .90 | .30 | .06 | .75 | .21 | .06 | .17 | .55 | .20 | .07 | .45 | .09 | .05 | .09 |
| | | $H_1^*$ | .96 | .41 | .24 | .75 | .49 | .24 | .53 | .70 | .21 | .18 | .43 | .28 | .15 | .33 |
| | | $H_2^*$ | .46 | .14 | .05 | .53 | .16 | .14 | .27 | .22 | .09 | .04 | .28 | .09 | .09 | .14 |
| | | $H_{max}^*$ | .89 | .32 | .08 | .75 | .29 | .15 | .40 | .53 | .16 | .06 | .42 | .13 | .10 | .21 |
| | | | $\alpha_1 = 0.2, \alpha_2 = 0.9, \alpha = 0.7$ | | | | | | | $\alpha_1 = 0.9, \alpha_2 = 0.2, \alpha = -0.7$ | | | | | | |
| | 3 | $H_1$ | .91 | .04 | .63 | .29 | .89 | .07 | .31 | .99 | .67 | .38 | .89 | .81 | .16 | .55 |
| | | $H_2$ | .44 | .03 | .22 | .25 | .31 | .04 | .09 | .64 | .34 | .09 | .66 | .32 | .08 | .15 |
| | | $H_{max}$ | .78 | .03 | .53 | .21 | .81 | .05 | .15 | .97 | .62 | .20 | .88 | .61 | .11 | .30 |
| | | $H_1^*$ | .81 | .03 | .47 | .29 | .71 | .16 | .36 | .95 | .44 | .40 | .76 | .76 | .23 | .57 |
| | | $H_2^*$ | .34 | .02 | .14 | .27 | .27 | .08 | .16 | .45 | .17 | .03 | .48 | .25 | .13 | .26 |
| | | $H_{max}^*$ | .69 | .03 | .38 | .27 | .64 | .11 | .23 | .89 | .42 | .16 | .75 | .61 | .17 | .42 |
| | | | $\alpha = 0.5$ | | | | | | | $\alpha = 0.7$ | | | | | | |
| 400 | 1 | $H_1$ | 1.0 | .07 | 1.0 | .64 | 1.0 | .26 | .70 | .98 | .08 | .92 | .13 | .97 | .09 | .33 |
| | | $H_2$ | .96 | .04 | .64 | .41 | .89 | .10 | .19 | .49 | .04 | .37 | .11 | .46 | .06 | .10 |
| | | $H_{max}$ | 1.0 | .05 | .97 | .57 | 1.0 | .19 | .42 | .93 | .05 | .75 | .11 | .90 | .07 | .15 |
| | | $H_1^*$ | 1.0 | .08 | .99 | .71 | 1.0 | .43 | .74 | .96 | .07 | .90 | .17 | .96 | .26 | .48 |
| | | $H_2^*$ | .91 | .05 | .63 | .44 | .89 | .27 | .45 | .46 | .05 | .38 | .15 | .48 | .14 | .20 |
| | | $H_{max}^*$ | 1.0 | .06 | .97 | .63 | 1.0 | .35 | .61 | .89 | .06 | .76 | .13 | .92 | .18 | .30 |
| | 2 | $H_1$ | 1.0 | 1.0 | .34 | 1.0 | .94 | .34 | .84 | 1.0 | .90 | .21 | .98 | .55 | .20 | .58 |
| | | $H_2$ | .96 | .67 | .05 | .98 | .29 | .12 | .27 | .64 | .45 | .04 | .78 | .12 | .08 | .13 |
| | | $H_{max}$ | 1.0 | .98 | .12 | 1.0 | .80 | .21 | .61 | .99 | .81 | .07 | .98 | .29 | .12 | .25 |
| | | $H_1^*$ | 1.0 | .99 | .37 | 1.0 | .92 | .50 | .87 | .99 | .87 | .25 | .97 | .62 | .34 | .68 |
| | | $H_2^*$ | .96 | .68 | .06 | .97 | .41 | .33 | .55 | .63 | .39 | .05 | .76 | .16 | .19 | .34 |
| | | $H_{max}^*$ | 1.0 | .97 | .14 | 1.0 | .83 | .41 | .76 | .98 | .77 | .09 | .96 | .37 | .25 | .52 |
| | | | $\alpha_1 = 0.2, \alpha_2 = 0.9, \alpha = 0.7$ | | | | | | | $\alpha_1 = 0.9, \alpha_2 = 0.2, \alpha = -0.7$ | | | | | | |
| | 3 | $H_1$ | 1.0 | .15 | .99 | .99 | 1.0 | .33 | .88 | 1.0 | .98 | .68 | 1.0 | 1.0 | .41 | .94 |
| | | $H_2$ | 1.0 | .05 | .86 | .93 | .99 | .12 | .36 | 1.0 | .92 | .17 | 1.0 | .89 | .19 | .47 |
| | | $H_{max}$ | 1.0 | .07 | .99 | .98 | 1.0 | .26 | .70 | 1.0 | .98 | .38 | 1.0 | 1.0 | .35 | .83 |
| | | $H_1^*$ | 1.0 | .15 | .97 | .97 | .99 | .44 | .84 | 1.0 | .95 | .72 | 1.0 | 1.0 | .52 | .92 |
| | | $H_2^*$ | .99 | .05 | .81 | .89 | .98 | .31 | .63 | .99 | .81 | .19 | .99 | .90 | .36 | .73 |
| | | $H_{max}^*$ | 1.0 | .07 | .95 | .96 | .99 | .39 | .78 | 1.0 | .94 | .47 | 1.0 | .99 | .45 | .87 |

Table B-5: Size-adjusted power [$m = 1$, $\rho = 0$, $\theta = 0.5$, $\lambda_1^0 = 0.5$]

| $T$ | DGP | Test | Model 1: $\delta$ | | | Model 2: $\delta$ | | Model 3: $c$ | | Model 1: $\delta$ | | | Model 2: $\delta$ | | Model 3: $c$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\delta/c$ | 1 | 1/3 | 3 | 1/3 | 3 | 0 | 10 | 1 | 1/3 | 3 | 1/3 | 3 | 0 | 10 |
| | | | $\alpha = 0.5$ | | | | | | | $\alpha = 0.7$ | | | | | | |
| 200 | 1 | $H_1$ | .84 | .03 | .85 | .12 | .90 | .07 | .26 | .53 | .03 | .51 | .03 | .63 | .03 | .08 |
| | | $H_2$ | .35 | .02 | .07 | .13 | .14 | .03 | .04 | .13 | .03 | .05 | .08 | .07 | .01 | .01 |
| | | $H_{max}$ | .71 | .02 | .56 | .09 | .73 | .04 | .10 | .37 | .04 | .23 | .03 | .39 | .02 | .03 |
| | | $H_1^*$ | .83 | .04 | .83 | .21 | .89 | .24 | .45 | .49 | .04 | .45 | .08 | .60 | .12 | .22 |
| | | $H_2^*$ | .34 | .04 | .12 | .20 | .31 | .09 | .20 | .11 | .04 | .07 | .08 | .10 | .05 | .09 |
| | | $H_{max}^*$ | .69 | .03 | .65 | .18 | .80 | .16 | .31 | .33 | .04 | .30 | .06 | .46 | .07 | .14 |
| | 2 | $H_1$ | .93 | .75 | .15 | .87 | .48 | .08 | .40 | .74 | .44 | .14 | .58 | .23 | .05 | .21 |
| | | $H_2$ | .47 | .12 | .02 | .47 | .10 | .04 | .06 | .23 | .07 | .03 | .25 | .05 | .04 | .04 |
| | | $H_{max}$ | .85 | .63 | .03 | .82 | .27 | .04 | .19 | .59 | .33 | .04 | .52 | .09 | .04 | .07 |
| | | $H_1^*$ | .91 | .77 | .19 | .86 | .56 | .28 | .58 | .66 | .41 | .12 | .53 | .29 | .16 | .34 |
| | | $H_2^*$ | .49 | .18 | .05 | .43 | .23 | .13 | .26 | .23 | .09 | .04 | .20 | .10 | .07 | .15 |
| | | $H_{max}^*$ | .84 | .66 | .09 | .83 | .41 | .20 | .47 | .56 | .33 | .05 | .49 | .19 | .10 | .24 |
| | | | $\alpha_1 = 0.2, \alpha_2 = 0.9, \alpha = 0.7$ | | | | | | | $\alpha_1 = 0.9, \alpha_2 = 0.2, \alpha = -0.7$ | | | | | | |
| | 3 | $H_1$ | .95 | .03 | .70 | .40 | .95 | .12 | .47 | .97 | .64 | .15 | .93 | .69 | .13 | .56 |
| | | $H_2$ | .72 | .01 | .07 | .44 | .39 | .04 | .07 | .77 | .10 | .01 | .69 | .27 | .04 | .08 |
| | | $H_{max}$ | .90 | .02 | .60 | .35 | .90 | .07 | .25 | .94 | .57 | .04 | .91 | .48 | .08 | .30 |
| | | $H_1^*$ | .93 | .04 | .48 | .41 | .87 | .25 | .58 | .95 | .44 | .11 | .82 | .66 | .26 | .60 |
| | | $H_2^*$ | .61 | .01 | .11 | .37 | .45 | .09 | .29 | .70 | .08 | .03 | .51 | .39 | .12 | .34 |
| | | $H_{max}^*$ | .86 | .02 | .40 | .40 | .83 | .18 | .48 | .91 | .41 | .05 | .81 | .58 | .20 | .52 |
| | | | $\alpha = 0.5$ | | | | | | | $\alpha = 0.7$ | | | | | | |
| 400 | 1 | $H_1$ | 1.0 | .05 | 1.0 | .64 | 1.0 | .37 | .68 | .97 | .05 | .97 | .19 | .99 | .14 | .39 |
| | | $H_2$ | .90 | .02 | .51 | .46 | .85 | .07 | .14 | .55 | .03 | .22 | .16 | .39 | .04 | .06 |
| | | $H_{max}$ | .99 | .03 | 1.0 | .61 | 1.0 | .26 | .45 | .90 | .03 | .90 | .17 | .94 | .07 | .17 |
| | | $H_1^*$ | 1.0 | .06 | 1.0 | .69 | 1.0 | .48 | .74 | .94 | .05 | .93 | .27 | .98 | .31 | .50 |
| | | $H_2^*$ | .84 | .04 | .61 | .54 | .84 | .27 | .41 | .44 | .05 | .25 | .20 | .47 | .14 | .20 |
| | | $H_{max}^*$ | .99 | .03 | 1.0 | .65 | 1.0 | .41 | .62 | .88 | .03 | .86 | .23 | .94 | .22 | .35 |
| | 2 | $H_1$ | 1.0 | 1.0 | .28 | 1.0 | .87 | .38 | .84 | .99 | .96 | .16 | .99 | .54 | .21 | .56 |
| | | $H_2$ | .95 | .60 | .04 | .95 | .32 | .09 | .19 | .70 | .31 | .02 | .73 | .13 | .06 | .10 |
| | | $H_{max}$ | 1.0 | 1.0 | .08 | 1.0 | .70 | .26 | .61 | .97 | .90 | .04 | .98 | .32 | .13 | .27 |
| | | $H_1^*$ | 1.0 | 1.0 | .37 | 1.0 | .90 | .54 | .85 | .98 | .92 | .21 | .98 | .59 | .36 | .66 |
| | | $H_2^*$ | .93 | .66 | .04 | .94 | .51 | .34 | .55 | .63 | .36 | .04 | .74 | .19 | .20 | .33 |
| | | $H_{max}^*$ | 1.0 | 1.0 | .13 | 1.0 | .80 | .46 | .75 | .95 | .87 | .07 | .97 | .40 | .28 | .52 |
| | | | $\alpha_1 = 0.2, \alpha_2 = 0.9, \alpha = 0.7$ | | | | | | | $\alpha_1 = 0.9, \alpha_2 = 0.2, \alpha = -0.7$ | | | | | | |
| | 3 | $H_1$ | 1.0 | .19 | .99 | .95 | 1.0 | .38 | .87 | 1.0 | .99 | .49 | 1.0 | .98 | .39 | .92 |
| | | $H_2$ | 1.0 | .05 | .90 | .93 | .98 | .12 | .38 | 1.0 | .89 | .06 | 1.0 | .87 | .14 | .42 |
| | | $H_{max}$ | 1.0 | .08 | .99 | .94 | 1.0 | .31 | .72 | 1.0 | .99 | .17 | 1.0 | .95 | .29 | .79 |
| | | $H_1^*$ | 1.0 | .23 | .97 | .95 | 1.0 | .44 | .86 | 1.0 | .95 | .47 | 1.0 | .98 | .49 | .93 |
| | | $H_2^*$ | .99 | .10 | .82 | .92 | .98 | .34 | .67 | 1.0 | .78 | .11 | .99 | .91 | .36 | .73 |
| | | $H_{max}^*$ | 1.0 | .11 | .97 | .94 | 1.0 | .41 | .80 | 1.0 | .95 | .23 | 1.0 | .96 | .44 | .87 |

Table B-6: Size-adjusted power [$m = 2$, $\rho = \theta = 0$, $\lambda_1^0 = 0.3$, $\lambda_2^0 = 0.8$]

| $T$ | DGP | Test | Model 1: $\delta$ | | | Model 2: $\delta$ | | Model 3: $c$ | | Model 1: $\delta$ | | | Model 2: $\delta$ | | Model 3: $c$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\delta/c$ | 1 | 1/3 | 3 | 1/3 | 3 | 0 | 10 | 1 | 1/3 | 3 | 1/3 | 3 | 0 | 10 |
| | | | $\alpha = 0.5$ | | | | | | | $\alpha = 0.7$ | | | | | | |
| 200 | 4 | $H_1$ | .38 | .12 | .43 | .14 | .38 | .05 | .13 | .27 | .02 | .26 | .07 | .24 | .04 | .07 |
| | | $H_2$ | .96 | .69 | .59 | .91 | .81 | .14 | .34 | .73 | .24 | .25 | .51 | .37 | .06 | .12 |
| | | $H_{max}$ | .95 | .54 | .56 | .72 | .74 | .07 | .31 | .66 | .15 | .23 | .24 | .32 | .04 | .11 |
| | | $H_1^*$ | .36 | .11 | .42 | .16 | .44 | .13 | .20 | .25 | .03 | .25 | .07 | .29 | .08 | .12 |
| | | $H_2^*$ | .95 | .62 | .61 | .89 | .83 | .26 | .57 | .68 | .22 | .25 | .42 | .44 | .11 | .25 |
| | | $H_{max}^*$ | .94 | .51 | .58 | .73 | .77 | .21 | .51 | .63 | .12 | .24 | .22 | .40 | .09 | .21 |
| | 5 | $H_1$ | .68 | .64 | .79 | .73 | .85 | .13 | .24 | .34 | .34 | .38 | .40 | .43 | .07 | .11 |
| | | $H_2$ | .46 | .05 | .06 | .40 | .14 | .05 | .04 | .08 | .04 | .03 | .11 | .04 | .02 | .03 |
| | | $H_{max}$ | .53 | .51 | .47 | .65 | .63 | .09 | .11 | .15 | .23 | .13 | .34 | .24 | .05 | .06 |
| | | $H_1^*$ | .73 | .65 | .82 | .72 | .87 | .28 | .41 | .34 | .35 | .38 | .40 | .45 | .15 | .22 |
| | | $H_2^*$ | .60 | .17 | .10 | .54 | .26 | .15 | .26 | .16 | .06 | .04 | .18 | .08 | .07 | .11 |
| | | $H_{max}^*$ | .65 | .59 | .52 | .68 | .68 | .23 | .31 | .23 | .28 | .17 | .35 | .25 | .11 | .15 |
| | | | $\alpha_1 = 0.2, \alpha_2 = 0.9, \alpha = 0.7$ | | | | | | | $\alpha_1 = 0.9, \alpha_2 = 0.2, \alpha = -0.7$ | | | | | | |
| | 6 | $H_1$ | .83 | .78 | .91 | .82 | .92 | .24 | .33 | .65 | .46 | .79 | .41 | .72 | .13 | .32 |
| | | $H_2$ | .86 | .25 | .21 | .78 | .49 | .03 | .07 | .99 | .85 | .79 | .95 | .92 | .25 | .60 |
| | | $H_{max}$ | .84 | .69 | .77 | .79 | .83 | .16 | .15 | .98 | .74 | .81 | .83 | .88 | .17 | .55 |
| | | $H_1^*$ | .77 | .74 | .88 | .75 | .90 | .29 | .44 | .50 | .38 | .69 | .34 | .64 | .20 | .27 |
| | | $H_2^*$ | .85 | .40 | .26 | .76 | .58 | .20 | .38 | .95 | .65 | .62 | .85 | .83 | .27 | .64 |
| | | $H_{max}^*$ | .81 | .70 | .78 | .74 | .84 | .25 | .37 | .91 | .56 | .70 | .68 | .75 | .21 | .51 |
| | | | $\alpha = 0.5$ | | | | | | | $\alpha = 0.7$ | | | | | | |
| 400 | 4 | $H_1$ | .49 | .30 | .60 | .26 | .50 | .17 | .19 | .46 | .20 | .44 | .22 | .43 | .09 | .15 |
| | | $H_2$ | 1.0 | .99 | .97 | 1.0 | 1.0 | .42 | .80 | .99 | .85 | .75 | .97 | .93 | .19 | .46 |
| | | $H_{max}$ | 1.0 | .98 | .97 | 1.0 | 1.0 | .35 | .77 | .98 | .75 | .74 | .89 | .91 | .13 | .39 |
| | | $H_1^*$ | .46 | .27 | .59 | .28 | .54 | .22 | .26 | .44 | .19 | .48 | .22 | .46 | .14 | .22 |
| | | $H_2^*$ | 1.0 | .99 | .96 | 1.0 | 1.0 | .56 | .91 | .99 | .79 | .73 | .95 | .94 | .33 | .69 |
| | | $H_{max}^*$ | 1.0 | .99 | .97 | 1.0 | .99 | .45 | .85 | .98 | .71 | .71 | .87 | .89 | .24 | .57 |
| | 5 | $H_1$ | 1.0 | .99 | 1.0 | .99 | 1.0 | .47 | .64 | .89 | .86 | .92 | .91 | .94 | .22 | .35 |
| | | $H_2$ | .99 | .28 | .47 | .94 | .86 | .07 | .16 | .68 | .05 | .11 | .50 | .30 | .04 | .04 |
| | | $H_{max}$ | .99 | .94 | .96 | .99 | 1.0 | .34 | .33 | .76 | .64 | .67 | .86 | .84 | .14 | .15 |
| | | $H_1^*$ | .99 | .98 | 1.0 | .99 | 1.0 | .66 | .77 | .91 | .84 | .92 | .89 | .95 | .41 | .51 |
| | | $H_2^*$ | .98 | .51 | .49 | .95 | .82 | .44 | .52 | .70 | .13 | .14 | .58 | .34 | .22 | .27 |
| | | $H_{max}^*$ | .99 | .96 | .97 | .98 | .99 | .55 | .61 | .77 | .71 | .68 | .86 | .84 | .31 | .34 |
| | | | $\alpha_1 = 0.2, \alpha_2 = 0.9, \alpha = 0.7$ | | | | | | | $\alpha_1 = 0.9, \alpha_2 = 0.2, \alpha = -0.7$ | | | | | | |
| | 6 | $H_1$ | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | .59 | .79 | .90 | .94 | .99 | .86 | .97 | .35 | .59 |
| | | $H_2$ | 1.0 | .82 | .91 | 1.0 | 1.0 | .19 | .48 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | .54 | .95 |
| | | $H_{max}$ | 1.0 | .99 | 1.0 | 1.0 | 1.0 | .51 | .59 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | .47 | .93 |
| | | $H_1^*$ | 1.0 | .99 | 1.0 | 1.0 | 1.0 | .70 | .84 | .79 | .88 | .96 | .73 | .91 | .46 | .60 |
| | | $H_2^*$ | 1.0 | .89 | .88 | 1.0 | .99 | .53 | .72 | 1.0 | 1.0 | .98 | 1.0 | 1.0 | .61 | .93 |
| | | $H_{max}^*$ | 1.0 | .99 | 1.0 | 1.0 | 1.0 | .64 | .77 | 1.0 | .98 | .98 | 1.0 | .99 | .55 | .87 |

Table B-7: Size-adjusted power [$m = 2$, $\rho = 0.5$, $\theta = 0$, $\lambda_1^0 = 0.3$, $\lambda_2^0 = 0.8$]

| $T$ | DGP | Test | Model 1: $\delta$ | | | Model 2: $\delta$ | | Model 3: $c$ | | Model 1: $\delta$ | | | Model 2: $\delta$ | | Model 3: $c$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\delta/c$ | 1 | 1/3 | 3 | 1/3 | 3 | 0 | 10 | 1 | 1/3 | 3 | 1/3 | 3 | 0 | 10 |
| | | | $\alpha = 0.5$ | | | | | | | $\alpha = 0.7$ | | | | | | |
| 200 | 4 | $H_1$ | .22 | .04 | .31 | .08 | .29 | .04 | .08 | .19 | .04 | .19 | .06 | .19 | .06 | .06 |
| | | $H_2$ | .89 | .43 | .49 | .74 | .65 | .10 | .22 | .58 | .20 | .23 | .36 | .32 | .06 | .09 |
| | | $H_{max}$ | .85 | .31 | .45 | .46 | .58 | .07 | .19 | .51 | .12 | .21 | .15 | .28 | .06 | .08 |
| | | $H_1^*$ | .21 | .05 | .29 | .08 | .28 | .09 | .11 | .16 | .03 | .17 | .06 | .20 | .06 | .08 |
| | | $H_2^*$ | .86 | .37 | .42 | .74 | .69 | .19 | .41 | .48 | .15 | .18 | .31 | .32 | .10 | .20 |
| | | $H_{max}^*$ | .80 | .30 | .38 | .49 | .56 | .13 | .29 | .40 | .10 | .15 | .14 | .26 | .07 | .14 |
| | 5 | $H_1$ | .51 | .51 | .56 | .58 | .67 | .11 | .19 | .28 | .30 | .29 | .35 | .33 | .07 | .12 |
| | | $H_2$ | .24 | .08 | .06 | .23 | .10 | .05 | .05 | .05 | .06 | .06 | .09 | .04 | .03 | .04 |
| | | $H_{max}$ | .31 | .40 | .26 | .51 | .44 | .08 | .10 | .10 | .21 | .12 | .27 | .16 | .05 | .07 |
| | | $H_1^*$ | .54 | .48 | .56 | .54 | .62 | .24 | .30 | .29 | .29 | .27 | .30 | .30 | .15 | .17 |
| | | $H_2^*$ | .26 | .12 | .08 | .34 | .14 | .15 | .20 | .10 | .07 | .05 | .14 | .06 | .07 | .10 |
| | | $H_{max}^*$ | .35 | .42 | .27 | .54 | .41 | .19 | .23 | .15 | .24 | .11 | .30 | .16 | .11 | .12 |
| | | | $\alpha_1 = 0.2, \alpha_2 = 0.9, \alpha = 0.7$ | | | | | | | $\alpha_1 = 0.9, \alpha_2 = 0.2, \alpha = -0.7$ | | | | | | |
| | 6 | $H_1$ | .68 | .68 | .67 | .71 | .72 | .18 | .25 | .36 | .07 | .62 | .07 | .51 | .07 | .14 |
| | | $H_2$ | .45 | .19 | .06 | .43 | .18 | .04 | .05 | .93 | .47 | .69 | .76 | .80 | .15 | .35 |
| | | $H_{max}$ | .52 | .59 | .33 | .65 | .53 | .13 | .12 | .90 | .32 | .67 | .46 | .72 | .09 | .30 |
| | | $H_1^*$ | .50 | .52 | .53 | .53 | .60 | .21 | .26 | .21 | .03 | .45 | .05 | .40 | .11 | .11 |
| | | $H_2^*$ | .41 | .18 | .07 | .41 | .23 | .14 | .18 | .67 | .26 | .42 | .54 | .63 | .17 | .38 |
| | | $H_{max}^*$ | .41 | .48 | .34 | .53 | .46 | .16 | .18 | .63 | .17 | .44 | .33 | .53 | .12 | .22 |
| | | | $\alpha = 0.5$ | | | | | | | $\alpha = 0.7$ | | | | | | |
| 400 | 4 | $H_1$ | .33 | .14 | .42 | .14 | .35 | .12 | .13 | .33 | .08 | .36 | .14 | .32 | .07 | .10 |
| | | $H_2$ | 1.0 | .96 | .94 | 1.0 | .99 | .29 | .64 | .97 | .62 | .63 | .92 | .86 | .13 | .31 |
| | | $H_{max}$ | 1.0 | .92 | .93 | .98 | .99 | .24 | .59 | .96 | .51 | .62 | .74 | .80 | .11 | .27 |
| | | $H_1^*$ | .28 | .12 | .40 | .15 | .38 | .19 | .18 | .29 | .09 | .35 | .12 | .34 | .12 | .16 |
| | | $H_2^*$ | 1.0 | .94 | .94 | 1.0 | .99 | .50 | .83 | .97 | .53 | .62 | .88 | .84 | .27 | .53 |
| | | $H_{max}^*$ | 1.0 | .91 | .90 | .98 | .97 | .38 | .72 | .94 | .45 | .59 | .66 | .74 | .21 | .42 |
| | 5 | $H_1$ | .98 | .96 | .98 | .97 | .99 | .37 | .53 | .75 | .76 | .77 | .79 | .84 | .19 | .28 |
| | | $H_2$ | .93 | .16 | .29 | .76 | .68 | .06 | .11 | .39 | .06 | .07 | .28 | .17 | .05 | .04 |
| | | $H_{max}$ | .95 | .86 | .89 | .96 | .97 | .25 | .28 | .53 | .54 | .44 | .74 | .65 | .11 | .11 |
| | | $H_1^*$ | .97 | .94 | .99 | .97 | 1.0 | .58 | .68 | .75 | .75 | .76 | .79 | .82 | .36 | .43 |
| | | $H_2^*$ | .89 | .26 | .34 | .80 | .64 | .38 | .43 | .43 | .10 | .10 | .34 | .21 | .20 | .23 |
| | | $H_{max}^*$ | .92 | .85 | .90 | .95 | .97 | .48 | .53 | .54 | .54 | .50 | .74 | .65 | .28 | .29 |
| | | | $\alpha_1 = 0.2, \alpha_2 = 0.9, \alpha = 0.7$ | | | | | | | $\alpha_1 = 0.9, \alpha_2 = 0.2, \alpha = -0.7$ | | | | | | |
| | 6 | $H_1$ | 1.0 | 1.0 | 1.0 | .99 | 1.0 | .50 | .69 | .59 | .55 | .92 | .41 | .81 | .21 | .31 |
| | | $H_2$ | .99 | .59 | .56 | .97 | .87 | .13 | .20 | 1.0 | .98 | .99 | 1.0 | 1.0 | .40 | .86 |
| | | $H_{max}$ | .99 | .98 | .97 | .99 | 1.0 | .40 | .42 | 1.0 | .93 | .99 | .99 | 1.0 | .33 | .80 |
| | | $H_1^*$ | .99 | .98 | .99 | .98 | 1.0 | .63 | .75 | .38 | .43 | .83 | .27 | .68 | .31 | .31 |
| | | $H_2^*$ | .97 | .63 | .58 | .92 | .88 | .43 | .55 | 1.0 | .92 | .95 | .99 | .99 | .52 | .86 |
| | | $H_{max}^*$ | .98 | .96 | .96 | .98 | .99 | .55 | .62 | 1.0 | .77 | .92 | .91 | .95 | .44 | .72 |

Table B-8: Size-adjusted power [$m = 2$, $\rho = 0$, $\theta = 0.5$, $\lambda_1^0 = 0.3$, $\lambda_2^0 = 0.8$]

| $T$ | DGP | Test | Model 1: $\delta$ | | | Model 2: $\delta$ | | Model 3: $c$ | | Model 1: $\delta$ | | | Model 2: $\delta$ | | Model 3: $c$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\delta/c$ | 1 | 1/3 | 3 | 1/3 | 3 | 0 | 10 | 1 | 1/3 | 3 | 1/3 | 3 | 0 | 10 |
| | | | $\alpha = 0.5$ | | | | | | | $\alpha = 0.7$ | | | | | | |
| 200 | 4 | $H_1$ | .40 | .15 | .38 | .15 | .37 | .05 | .13 | .34 | .06 | .23 | .11 | .24 | .03 | .08 |
| | | $H_2$ | .89 | .47 | .36 | .77 | .60 | .09 | .24 | .63 | .18 | .16 | .46 | .28 | .05 | .09 |
| | | $H_{max}$ | .86 | .38 | .36 | .56 | .54 | .05 | .21 | .61 | .13 | .16 | .26 | .25 | .03 | .08 |
| | | $H_1^*$ | .40 | .14 | .39 | .18 | .46 | .13 | .20 | .27 | .06 | .23 | .09 | .27 | .07 | .13 |
| | | $H_2^*$ | .87 | .47 | .46 | .77 | .73 | .20 | .48 | .55 | .17 | .18 | .36 | .35 | .09 | .21 |
| | | $H_{max}^*$ | .86 | .36 | .42 | .65 | .69 | .15 | .38 | .51 | .11 | .18 | .23 | .33 | .06 | .16 |
| | 5 | $H_1$ | .27 | .33 | .56 | .33 | .57 | .08 | .13 | .15 | .20 | .33 | .20 | .29 | .04 | .06 |
| | | $H_2$ | .26 | .03 | .02 | .22 | .07 | .02 | .02 | .06 | .02 | .02 | .07 | .03 | .02 | .01 |
| | | $H_{max}$ | .25 | .22 | .28 | .27 | .31 | .04 | .05 | .08 | .12 | .11 | .15 | .11 | .03 | .02 |
| | | $H_1^*$ | .26 | .35 | .55 | .31 | .56 | .17 | .19 | .15 | .22 | .31 | .18 | .30 | .10 | .13 |
| | | $H_2^*$ | .34 | .09 | .07 | .33 | .16 | .13 | .21 | .13 | .05 | .03 | .14 | .06 | .07 | .10 |
| | | $H_{max}^*$ | .30 | .30 | .34 | .29 | .40 | .13 | .16 | .14 | .16 | .17 | .16 | .18 | .08 | .09 |
| | | | $\alpha_1 = 0.2, \alpha_2 = 0.9, \alpha = 0.7$ | | | | | | | $\alpha_1 = 0.9, \alpha_2 = 0.2, \alpha = -0.7$ | | | | | | |
| | 6 | $H_1$ | .35 | .40 | .71 | .37 | .69 | .11 | .13 | .75 | .51 | .70 | .50 | .70 | .11 | .32 |
| | | $H_2$ | .54 | .11 | .11 | .46 | .24 | .02 | .03 | .95 | .64 | .54 | .86 | .76 | .15 | .35 |
| | | $H_{max}$ | .50 | .31 | .46 | .33 | .49 | .07 | .06 | .93 | .57 | .56 | .71 | .75 | .10 | .32 |
| | | $H_1^*$ | .25 | .28 | .53 | .22 | .56 | .16 | .17 | .51 | .37 | .54 | .36 | .58 | .17 | .30 |
| | | $H_2^*$ | .54 | .16 | .16 | .40 | .29 | .11 | .26 | .80 | .38 | .39 | .61 | .53 | .17 | .45 |
| | | $H_{max}^*$ | .40 | .25 | .43 | .24 | .46 | .13 | .18 | .71 | .40 | .46 | .50 | .59 | .17 | .39 |
| | | | $\alpha = 0.5$ | | | | | | | $\alpha = 0.7$ | | | | | | |
| 400 | 4 | $H_1$ | .52 | .35 | .59 | .29 | .53 | .16 | .21 | .47 | .24 | .44 | .23 | .44 | .08 | .16 |
| | | $H_2$ | 1.0 | .94 | .85 | .99 | .97 | .31 | .63 | .98 | .63 | .53 | .91 | .81 | .14 | .34 |
| | | $H_{max}$ | 1.0 | .91 | .86 | .98 | .96 | .23 | .59 | .96 | .56 | .53 | .80 | .78 | .10 | .30 |
| | | $H_1^*$ | .47 | .34 | .60 | .31 | .57 | .26 | .30 | .38 | .21 | .40 | .23 | .42 | .17 | .23 |
| | | $H_2^*$ | 1.0 | .92 | .88 | .99 | .98 | .51 | .82 | .94 | .59 | .53 | .87 | .78 | .31 | .58 |
| | | $H_{max}^*$ | 1.0 | .89 | .86 | .98 | .96 | .42 | .74 | .90 | .49 | .46 | .73 | .69 | .25 | .47 |
| | 5 | $H_1$ | .77 | .80 | .95 | .80 | .95 | .37 | .37 | .53 | .57 | .79 | .62 | .77 | .20 | .21 |
| | | $H_2$ | .85 | .09 | .24 | .68 | .54 | .04 | .07 | .42 | .02 | .05 | .28 | .15 | .03 | .03 |
| | | $H_{max}$ | .85 | .63 | .82 | .77 | .88 | .25 | .17 | .43 | .36 | .49 | .54 | .55 | .11 | .07 |
| | | $H_1^*$ | .71 | .82 | .93 | .82 | .94 | .52 | .46 | .46 | .59 | .76 | .58 | .76 | .33 | .30 |
| | | $H_2^*$ | .81 | .25 | .28 | .77 | .59 | .38 | .41 | .40 | .08 | .07 | .42 | .21 | .21 | .25 |
| | | $H_{max}^*$ | .81 | .68 | .81 | .80 | .87 | .47 | .41 | .42 | .42 | .47 | .55 | .59 | .27 | .22 |
| | | | $\alpha_1 = 0.2, \alpha_2 = 0.9, \alpha = 0.7$ | | | | | | | $\alpha_1 = 0.9, \alpha_2 = 0.2, \alpha = -0.7$ | | | | | | |
| | 6 | $H_1$ | .89 | .91 | .99 | .87 | .98 | .42 | .41 | .97 | .96 | .98 | .91 | .98 | .33 | .69 |
| | | $H_2$ | .98 | .50 | .68 | .93 | .92 | .09 | .23 | 1.0 | .99 | .96 | 1.0 | .99 | .42 | .84 |
| | | $H_{max}$ | .98 | .82 | .96 | .89 | .97 | .33 | .29 | 1.0 | .98 | .97 | .99 | .99 | .36 | .82 |
| | | $H_1^*$ | .80 | .88 | .97 | .82 | .98 | .49 | .53 | .82 | .89 | .94 | .83 | .92 | .46 | .64 |
| | | $H_2^*$ | .98 | .63 | .68 | .95 | .93 | .42 | .57 | 1.0 | .94 | .90 | .99 | .97 | .52 | .87 |
| | | $H_{max}^*$ | .96 | .80 | .94 | .87 | .97 | .45 | .54 | .99 | .92 | .93 | .96 | .96 | .49 | .79 |

Table B-9: Break selection probabilities, $[\rho = \theta = 0,\ \eta = .10]$

| T | m | $\lambda_1^0/\lambda_1^0, \lambda_2^0$ | DGP | $\alpha/(\alpha_1, \alpha_2)$ | | Model 1: $\delta$ | | | Model 2: $\delta$ | | Model 3: $c$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | $\delta/c$ | 1 | 1/3 | 3 | 1/3 | 3 | 0 | 10 |
| 400 | 0 | | 0 | 1 | $P_c$ | .91 | .91 | .91 | .92 | .90 | .90 | .91 |
| | | | | | $P_o$ | .09 | .09 | .09 | .08 | .10 | .10 | .10 |
| | | | | 0.5 | $P_c$ | .86 | .91 | .90 | .88 | .89 | .91 | .90 |
| | | | | | $P_o$ | .14 | .09 | .10 | .12 | .11 | .10 | .10 |
| | 1 | $\lambda_1^0 = 0.5$ | 1 | 0.5 | $P_c$ | .90 | .07 | .91 | .82 | .89 | .39 | .68 |
| | | | | | $P_o$ | .10 | .04 | .09 | .14 | .11 | .15 | .14 |
| | | | 2 | 0.5 | $P_c$ | .88 | .87 | .21 | .88 | .79 | .47 | .73 |
| | | | | | $P_o$ | .12 | .12 | .12 | .12 | .20 | .14 | .15 |
| | | | 3 | (0.2, 0.9) | $P_c$ | .89 | .42 | .87 | .88 | .89 | .46 | .80 |
| | | | | | $P_o$ | .11 | .08 | .13 | .12 | .11 | .13 | .13 |
| | | | | (0.9, 0.2) | $P_c$ | .88 | .82 | .62 | .88 | .85 | .48 | .81 |
| | | | | | $P_o$ | .12 | .18 | .14 | .12 | .15 | .13 | .12 |
| | 2 | $\lambda_1^0 = 0.3, \lambda_2^0 = 0.8$ | 4 | 0.5 | $P_c$ | .86 | .70 | .46 | .85 | .70 | .27 | .59 |
| | | | | | $P_o$ | .10 | .17 | .16 | .11 | .13 | .11 | .14 |
| | | | 5 | 0.5 | $P_c$ | .83 | .07 | .08 | .39 | .28 | .18 | .36 |
| | | | | | $P_o$ | .15 | .08 | .06 | .12 | .18 | .09 | .10 |
| | | | 6 | (0.2, 0.9) | $P_c$ | .89 | .08 | .11 | .84 | .73 | .24 | .57 |
| | | | | | $P_o$ | .12 | .07 | .09 | .12 | .16 | .09 | .11 |
| | | | | (0.9, 0.2) | $P_c$ | .90 | .72 | .62 | .89 | .84 | .32 | .75 |
| | | | | | $P_o$ | .09 | .23 | .20 | .10 | .12 | .12 | .11 |
| 600 | 0 | | 0 | 1 | $P_c$ | .89 | .88 | .89 | .89 | .89 | .88 | .89 |
| | | | | | $P_o$ | .11 | .12 | .11 | .11 | .11 | .12 | .11 |
| | | | | 0.5 | $P_c$ | .87 | .89 | .88 | .88 | .88 | .91 | .89 |
| | | | | | $P_o$ | .13 | .11 | .12 | .12 | .12 | .09 | .11 |
| | 1 | $\lambda_1^0 = 0.5$ | 1 | 0.5 | $P_c$ | .88 | .11 | .89 | .87 | .89 | .49 | .80 |
| | | | | | $P_o$ | .12 | .05 | .11 | .13 | .11 | .17 | .11 |
| | | | 2 | 0.5 | $P_c$ | .89 | .88 | .37 | .89 | .86 | .51 | .79 |
| | | | | | $P_o$ | .11 | .12 | .21 | .11 | .14 | .17 | .15 |
| | | | 3 | (0.2, 0.9) | $P_c$ | .85 | .71 | .84 | .85 | .85 | .56 | .86 |
| | | | | | $P_o$ | .15 | .17 | .17 | .15 | .15 | .12 | .11 |
| | | | | (0.9, 0.2) | $P_c$ | .83 | .82 | .75 | .86 | .82 | .55 | .85 |
| | | | | | $P_o$ | .17 | .18 | .22 | .14 | .18 | .14 | .12 |
| | 2 | $\lambda_1^0 = 0.3, \lambda_2^0 = 0.8$ | 4 | 0.5 | $P_c$ | .86 | .76 | .64 | .86 | .78 | .34 | .69 |
| | | | | | $P_o$ | .11 | .20 | .15 | .12 | .12 | .14 | .16 |
| | | | 5 | 0.5 | $P_c$ | .87 | .05 | .08 | .76 | .53 | .27 | .51 |
| | | | | | $P_o$ | .13 | .07 | .08 | .13 | .22 | .10 | .13 |
| | | | 6 | (0.2, 0.9) | $P_c$ | .86 | .33 | .30 | .85 | .84 | .33 | .69 |
| | | | | | $P_o$ | .14 | .12 | .13 | .15 | .16 | .11 | .13 |
| | | | | (0.9, 0.2) | $P_c$ | .90 | .83 | .78 | .91 | .89 | .38 | .81 |
| | | | | | $P_o$ | .10 | .17 | .18 | .09 | .11 | .14 | .13 |

Table B-10: Break selection probabilities [Model 1, DGP 5 and 6, $m = 2$, $\rho = \theta = 0$]

| T | DGP | $\alpha/(\alpha_1, \alpha_2)$ | $\tau$ | | Model 1: $\delta$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | 1/3 | 1/2.5 | 1/1.5 | 1/1.1 | 1 | 1.1 | 1.5 | 2.5 | 3 |
| 400 | 5 | 0.5 | 0.2 | $P_c$ | .47 | .54 | .76 | .83 | .84 | .84 | .84 | .63 | .44 |
| | | | | $P_o$ | .13 | .15 | .17 | .16 | .15 | .14 | .14 | .30 | .45 |
| | | | 0.3 | $P_c$ | .11 | .11 | .62 | .83 | .84 | .85 | .80 | .50 | .35 |
| | | | | $P_o$ | .06 | .07 | .14 | .15 | .14 | .14 | .16 | .37 | .46 |
| | | | 0.5 | $P_c$ | .07 | .06 | .66 | .84 | .84 | .82 | .66 | .08 | .08 |
| | | | | $P_o$ | .08 | .10 | .14 | .14 | .15 | .16 | .17 | .08 | .06 |
| | | | 0.8 | $P_c$ | .68 | .77 | .87 | .86 | .84 | .80 | .61 | .18 | .15 |
| | | | | $P_o$ | .15 | .13 | .12 | .13 | .15 | .17 | .19 | .08 | .06 |
| | | | 0.9 | $P_c$ | .89 | .88 | .87 | .85 | .84 | .83 | .78 | .69 | .66 |
| | | | | $P_o$ | .09 | .11 | .12 | .14 | .15 | .15 | .17 | .17 | .16 |
| | 6 | (0.2, 0.9) | 0.2 | $P_c$ | .86 | .85 | .89 | .89 | .90 | .90 | .89 | .72 | .55 |
| | | | | $P_o$ | .12 | .14 | .11 | .11 | .11 | .10 | .11 | .27 | .42 |
| | | | 0.3 | $P_c$ | .18 | .41 | .88 | .89 | .89 | .90 | .88 | .67 | .50 |
| | | | | $P_o$ | .07 | .09 | .11 | .11 | .11 | .10 | .12 | .29 | .40 |
| | | | 0.5 | $P_c$ | .08 | .28 | .88 | .88 | .90 | .89 | .88 | .31 | .11 |
| | | | | $P_o$ | .07 | .09 | .12 | .12 | .10 | .11 | .11 | .12 | .09 |
| | | | 0.8 | $P_c$ | .80 | .86 | .88 | .89 | .88 | .89 | .86 | .46 | .32 |
| | | | | $P_o$ | .14 | .12 | .12 | .11 | .12 | .11 | .12 | .12 | .08 |
| | | | 0.9 | $P_c$ | .90 | .90 | .90 | .89 | .89 | .88 | .88 | .86 | .86 |
| | | | | $P_o$ | .09 | .10 | .10 | .11 | .11 | .12 | .12 | .13 | .13 |
| 600 | 5 | 0.5 | 0.2 | $P_c$ | .78 | .80 | .85 | .86 | .87 | .87 | .88 | .74 | .58 |
| | | | | $P_o$ | .15 | .16 | .15 | .14 | .13 | .13 | .12 | .25 | .41 |
| | | | 0.3 | $P_c$ | .09 | .12 | .83 | .86 | .87 | .87 | .88 | .71 | .54 |
| | | | | $P_o$ | .05 | .06 | .15 | .14 | .13 | .13 | .12 | .27 | .40 |
| | | | 0.5 | $P_c$ | .05 | .07 | .86 | .88 | .86 | .85 | .83 | .13 | .07 |
| | | | | $P_o$ | .07 | .08 | .13 | .12 | .14 | .15 | .16 | .10 | .08 |
| | | | 0.8 | $P_c$ | .82 | .86 | .87 | .87 | .86 | .86 | .76 | .25 | .18 |
| | | | | $P_o$ | .15 | .13 | .13 | .14 | .14 | .14 | .21 | .13 | .09 |
| | | | 0.9 | $P_c$ | .88 | .89 | .88 | .86 | .87 | .87 | .85 | .81 | .81 |
| | | | | $P_o$ | .12 | .11 | .12 | .14 | .14 | .14 | .15 | .18 | .18 |
| | 6 | (0.2, 0.9) | 0.2 | $P_c$ | .84 | .85 | .86 | .85 | .86 | .86 | .86 | .80 | .67 |
| | | | | $P_o$ | .16 | .15 | .15 | .15 | .14 | .14 | .14 | .20 | .33 |
| | | | 0.3 | $P_c$ | .57 | .76 | .86 | .86 | .86 | .86 | .87 | .79 | .67 |
| | | | | $P_o$ | .12 | .15 | .14 | .14 | .14 | .14 | .13 | .21 | .32 |
| | | | 0.5 | $P_c$ | .32 | .68 | .85 | .86 | .85 | .86 | .88 | .64 | .30 |
| | | | | $P_o$ | .11 | .15 | .15 | .14 | .15 | .14 | .12 | .14 | .13 |
| | | | 0.8 | $P_c$ | .84 | .85 | .86 | .86 | .86 | .87 | .86 | .74 | .56 |
| | | | | $P_o$ | .16 | .16 | .14 | .14 | .14 | .14 | .14 | .15 | .12 |
| | | | 0.9 | $P_c$ | .86 | .86 | .86 | .86 | .85 | .86 | .86 | .87 | .85 |
| | | | | $P_o$ | .14 | .14 | .14 | .14 | .15 | .14 | .14 | .14 | .15 |

Table B-11: Model selection probabilities between pure mean shift and persistence break [$\eta = .10$]

| T | DGP | $\Delta\mu/\alpha/\alpha_1, \alpha_2$ | Model 1: $\delta$ | | | Model 2: $\delta$ | | Model 3: $c$ | |
|---|---|---|---|---|---|---|---|---|---|
| | | | 1 | 1/3 | 3 | 1/3 | 3 | 0 | 10 |
| 200 | $0_\mu$ | $\Delta\mu = 1$ | .87 | .32 | .83 | .38 | .84 | .31 | .63 |
| | | $\Delta\mu = 3$ | .86 | .84 | .86 | .88 | .84 | .48 | .77 |
| | 1 | $\alpha = 0.5$ | .97 | .05 | .94 | .35 | .98 | .33 | .62 |
| | 2 | $\alpha = 0.5$ | .98 | .94 | .26 | .99 | .68 | .42 | .78 |
| | 3 | $\alpha_1 = 0.2, \alpha_2 = 0.9$ | 1.00 | .12 | .93 | .89 | .99 | .47 | .86 |
| | | $\alpha_1 = 0.9, \alpha_2 = 0.2$ | 1.00 | .93 | .48 | .99 | .96 | .51 | .91 |
| 400 | $0_\mu$ | $\Delta\mu = 1$ | .87 | .62 | .87 | .74 | .89 | .48 | .82 |
| | | $\Delta\mu = 3$ | .89 | .86 | .87 | .89 | .86 | .63 | .88 |
| | 1 | $\alpha = 0.5$ | 1.00 | .06 | 1.00 | .96 | 1.00 | .59 | .87 |
| | 2 | $\alpha = 0.5$ | 1.00 | 1.00 | .54 | 1.00 | .99 | .64 | .95 |
| | 3 | $\alpha_1 = 0.2, \alpha_2 = 0.9$ | 1.00 | .65 | 1.00 | 1.00 | 1.00 | .62 | .97 |
| | | $\alpha_1 = 0.9, \alpha_2 = 0.2$ | 1.00 | 1.00 | .96 | 1.00 | 1.00 | .61 | .99 |
| 600 | $0_\mu$ | $\Delta\mu = 1$ | .90 | .75 | .88 | .84 | .90 | .51 | .87 |
| | | $\Delta\mu = 3$ | .90 | .87 | .87 | .87 | .89 | .69 | .89 |
| | 1 | $\alpha = 0.5$ | 1.00 | .15 | 1.00 | 1.00 | 1.00 | .64 | .93 |
| | 2 | $\alpha = 0.5$ | 1.00 | 1.00 | .85 | 1.00 | 1.00 | .70 | .98 |
| | 3 | $\alpha_1 = 0.2, \alpha_2 = 0.9$ | 1.00 | .99 | 1.00 | 1.00 | 1.00 | .70 | .98 |
| | | $\alpha_1 = 0.9, \alpha_2 = 0.2$ | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | .69 | 1.00 |

## Supplement C: Comparison with the Recursive Bootstrap

In order to highlight the advantages of employing the proposed bootstrap schemes A and B, we now provide a comparison with the fully recursive bootstrap schemes. The recursive counterpart of scheme A entails replacing step 3 in scheme A with the recursion

$$\begin{aligned} y_t^{(1)} &= y_{t-1}^{(1)} + \textstyle\sum_{j=1}^{\breve{l}_T} \breve{\pi}_j \Delta y_{t-j}^{(1)} + u_t^{(1)}; \quad t = \breve{l}_T + 2, ..., T \\ y_t^{(1)} &= y_t; \ t = 1, ..., \breve{l}_T + 1 \end{aligned} \tag{C.1}$$

while the recursive counterpart of scheme B involves replacing step 3 in scheme B with the recursion

$$\begin{aligned} y_t^{(0)} &= \widetilde{c} + \widetilde{\alpha} y_{t-1}^{(0)} + \textstyle\sum_{j=1}^{\widetilde{l}_T} \widetilde{\pi}_j \Delta y_{t-j}^{(0)} + u_t^{(0)}; \quad t = \breve{l}_T + 2, ..., T \\ y_t^{(0)} &= 0; \ t = 1, ..., \widetilde{l}_T + 1 \end{aligned} \tag{C.2}$$

Since the bootstrap data obtained from $(C.1)$ and $(C.2)$ are serially correlated, conditional on the original data, the bootstrap statistics will now need to be adjusted by including lagged first differences in the estimated regression as in the construction of the statistics based on the original data $\{y_t\}$. The lag length is again chosen using the BIC. Table C-1 reports the empirical size and size-adjusted power (only in the single break case, for brevity) of the recursive bootstrap tests (denoted with a superscript "$r$") for $\rho = \theta = 0$. The procedure has accurate size in general with a tendency to under-reject in some cases. A power comparison with Table B-2 reveals that the recursive bootstrap tests are generally less powerful than the hybrid tests for DGP-1 and DGP-2 which contain an $I(1)$ segment, in accordance with the discussion in Section 5. For DGP-3, the two approaches yield comparable power. The power gains are even more transparent if one were to *a priori* rule out the $I(1)$ null hypothesis and hence apply the BP tests in isolation (see Tables C-2 and C-3). Overall, these findings favor the use of the proposed scheme over the recursive scheme in terms of its relative ability in detecting persistence change.

Table C-1: Size and size-adjusted power of bootstrap recursive tests, $[\rho = \theta = 0,\ \lambda_1^0 = 0.5,\ 5\%]$

| $T$ | DGP | Test | Model 1: $\delta$ | | | Model 2: $\delta$ | | Model 3: $c$ | | Model 1: $\delta$ | | | Model 2: $\delta$ | | Model 3: $c$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\delta/c$ | 1 | 1/3 | 3 | 1/3 | 3 | 0 | 10 | 1 | 1/3 | 3 | 1/3 | 3 | 0 | 10 |
| | | | $\alpha = 1$ | | | | | | | $\alpha = 0.5$ | | | | | | |
| 200 | 0 | $H_1^r$ | .03 | .05 | .04 | .05 | .03 | .06 | .06 | .04 | .03 | .04 | .04 | .03 | .03 | .03 |
| | | $H_2^r$ | .02 | .02 | .03 | .02 | .02 | .05 | .03 | .03 | .02 | .01 | .04 | .02 | .03 | .03 |
| | | $H_{max}^r$ | .02 | .04 | .02 | .05 | .02 | .05 | .04 | .04 | .02 | .02 | .04 | .02 | .03 | .03 |
| | | | $\alpha = 0.5$ | | | | | | | $\alpha = 0.7$ | | | | | | |
| | 1 | $H_1^r$ | .95 | .05 | .74 | .26 | .92 | .25 | .51 | .57 | .04 | .36 | .06 | .54 | .08 | .20 |
| | | $H_2^r$ | .53 | .04 | .18 | .25 | .38 | .11 | .21 | .15 | .04 | .07 | .06 | .14 | .05 | .07 |
| | | $H_{max}^r$ | .89 | .04 | .59 | .21 | .84 | .17 | .37 | .41 | .04 | .22 | .05 | .42 | .07 | .11 |
| | 2 | $H_1^r$ | .99 | .74 | .26 | .92 | .70 | .30 | .69 | .79 | .40 | .16 | .57 | .28 | .16 | .39 |
| | | $H_2^r$ | .66 | .29 | .05 | .65 | .22 | .14 | .29 | .26 | .14 | .04 | .30 | .09 | .08 | .12 |
| | | $H_{max}^r$ | .97 | .65 | .10 | .90 | .50 | .23 | .57 | .69 | .31 | .06 | .53 | .16 | .12 | .26 |
| | | | $\alpha_1 = 0.2, \alpha_2 = 0.9, \alpha = 0.7$ | | | | | | | $\alpha_1 = 0.9, \alpha_2 = 0.2, \alpha = -0.7$ | | | | | | |
| | 3 | $H_1^r$ | .99 | .08 | .82 | .74 | .97 | .34 | .72 | 1.0 | .88 | .33 | .99 | .91 | .42 | .82 |
| | | $H_2^r$ | .87 | .05 | .32 | .64 | .73 | .19 | .43 | .93 | .45 | .06 | .89 | .59 | .24 | .51 |
| | | $H_{max}^r$ | .99 | .04 | .77 | .69 | .97 | .30 | .64 | 1.0 | .85 | .17 | .99 | .85 | .36 | .73 |
| | | | $\alpha = 1$ | | | | | | | $\alpha = 0.5$ | | | | | | |
| 400 | 0 | $H_1^r$ | .03 | .04 | .05 | .04 | .04 | .05 | .05 | .05 | .05 | .04 | .05 | .05 | .05 | .06 |
| | | $H_2^r$ | .02 | .03 | .03 | .02 | .02 | .04 | .03 | .06 | .03 | .03 | .04 | .04 | .03 | .04 |
| | | $H_{max}^r$ | .02 | .02 | .02 | .04 | .02 | .05 | .03 | .05 | .04 | .04 | .05 | .04 | .05 | .05 |
| | | | $\alpha = 0.5$ | | | | | | | $\alpha = 0.7$ | | | | | | |
| | 1 | $H_1^r$ | 1.0 | .07 | 1.0 | .94 | 1.0 | .50 | .85 | 1.0 | .06 | .91 | .31 | .98 | .31 | .61 |
| | | $H_2^r$ | .99 | .03 | .72 | .77 | .95 | .30 | .59 | .68 | .03 | .42 | .22 | .60 | .14 | .28 |
| | | $H_{max}^r$ | 1.0 | .04 | 1.0 | .91 | 1.0 | .41 | .76 | .98 | .03 | .83 | .23 | .96 | .22 | .42 |
| | 2 | $H_1^r$ | 1.0 | 1.0 | .48 | 1.0 | .99 | .58 | .93 | 1.0 | .92 | .25 | .99 | .74 | .39 | .78 |
| | | $H_2^r$ | .99 | .77 | .06 | .99 | .65 | .39 | .68 | .82 | .54 | .05 | .88 | .22 | .22 | .40 |
| | | $H_{max}^r$ | 1.0 | 1.0 | .20 | 1.0 | .96 | .50 | .85 | 1.0 | .89 | .10 | .99 | .54 | .29 | .63 |
| | | | $\alpha_1 = 0.2, \alpha_2 = 0.9, \alpha = 0.7$ | | | | | | | $\alpha_1 = 0.9, \alpha_2 = 0.2, \alpha = -0.7$ | | | | | | |
| | 3 | $H_1^r$ | 1.0 | .47 | 1.0 | 1.0 | 1.0 | .55 | .94 | 1.0 | 1.0 | .82 | 1.0 | 1.0 | .60 | .97 |
| | | $H_2^r$ | 1.0 | .16 | .96 | .99 | 1.0 | .46 | .85 | 1.0 | .96 | .26 | 1.0 | .98 | .48 | .87 |
| | | $H_{max}^r$ | 1.0 | .30 | 1.0 | 1.0 | 1.0 | .52 | .92 | 1.0 | 1.0 | .55 | 1.0 | 1.0 | .55 | .93 |

Table C-2: Empirical power of bootstrap recursive and non-recursive $G_1$ tests, $[m = 1, \theta = 0, \lambda_1^0 = 0.5]$

| T | $\rho$ | DGP | Test | Model 1: $\delta$ | | | Model 2: $\delta$ | | Model 3: c | | Model 1: $\delta$ | | | Model 2: $\delta$ | | Model 3: c | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $\delta/c$ | 1 | 1/3 | 3 | 1/3 | 3 | 0 | 10 | 1 | 1/3 | 3 | 1/3 | 3 | 0 | 10 |
| | | | | $\alpha = 0.5\ /\alpha_1 = 0.2, \alpha_2 = 0.9$ | | | | | | | $\alpha = 0.7/\alpha_1 = 0.9, \alpha_2 = 0.2$ | | | | | | |
| 200 | 0.5 | 1 | $G_1^r$ | .89 | .20 | .60 | .42 | .82 | .37 | .62 | .57 | .16 | .31 | .23 | .48 | .27 | .37 |
| | | | $G_1^*$ | .98 | .72 | .86 | .88 | .97 | .63 | .88 | .90 | .66 | .64 | .71 | .83 | .52 | .73 |
| | | 2 | $G_1^r$ | .94 | .50 | .43 | .76 | .66 | .44 | .72 | .68 | .26 | .35 | .45 | .42 | .32 | .51 |
| | | | $G_1^*$ | .99 | .82 | .83 | .96 | .92 | .65 | .91 | .93 | .61 | .74 | .83 | .79 | .55 | .79 |
| | | 3 | $G_1^r$ | .90 | .14 | .63 | .56 | .83 | .37 | .65 | .98 | .60 | .59 | .87 | .88 | .49 | .82 |
| | | | $G_1^*$ | .97 | .42 | .74 | .85 | .92 | .48 | .78 | .99 | .77 | .78 | .95 | .96 | .56 | .90 |
| | 0.8 | 1 | $G_1^r$ | .62 | .20 | .36 | .34 | .55 | .31 | .45 | .40 | .17 | .17 | .24 | .29 | .23 | .29 |
| | | | $G_1^*$ | .94 | .74 | .80 | .81 | .90 | .61 | .79 | .83 | .70 | .62 | .71 | .77 | .53 | .67 |
| | | 2 | $G_1^r$ | .74 | .28 | .40 | .54 | .49 | .37 | .55 | .53 | .15 | .36 | .29 | .38 | .27 | .40 |
| | | | $G_1^*$ | .96 | .75 | .82 | .91 | .86 | .62 | .85 | .91 | .55 | .78 | .77 | .77 | .54 | .76 |
| | | 3 | $G_1^r$ | .64 | .10 | .37 | .21 | .58 | .26 | .44 | .90 | .38 | .72 | .74 | .74 | .43 | .72 |
| | | | $G_1^*$ | .87 | .38 | .57 | .60 | .82 | .43 | .63 | .96 | .65 | .84 | .92 | .88 | .56 | .83 |
| | | | | $\alpha = 0.5\ /\alpha_1 = 0.2, \alpha_2 = 0.9$ | | | | | | | $\alpha = 0.7/\alpha_1 = 0.9, \alpha_2 = 0.2$ | | | | | | |
| 400 | 0.5 | 1 | $G_1^r$ | 1.0 | .24 | .99 | .86 | 1.0 | .57 | .86 | .97 | .16 | .89 | .40 | .96 | .42 | .70 |
| | | | $G_1^*$ | 1.0 | .81 | 1.0 | .99 | 1.0 | .78 | .97 | .99 | .70 | .96 | .91 | .99 | .67 | .91 |
| | | 2 | $G_1^r$ | 1.0 | .99 | .61 | 1.0 | .97 | .68 | .95 | .99 | .84 | .47 | .95 | .72 | .53 | .82 |
| | | | $G_1^*$ | 1.0 | 1.0 | .95 | 1.0 | 1.0 | .79 | .98 | 1.0 | .95 | .87 | .99 | .95 | .71 | .94 |
| | | 3 | $G_1^r$ | 1.0 | .36 | .98 | .99 | 1.0 | .58 | .95 | 1.0 | .98 | .85 | 1.0 | 1.0 | .64 | .97 |
| | | | $G_1^*$ | 1.0 | .75 | .99 | 1.0 | 1.0 | .63 | .97 | 1.0 | .99 | .94 | 1.0 | 1.0 | .67 | .98 |
| | 0.8 | 1 | $G_1^r$ | .94 | .20 | .89 | .47 | .95 | .45 | .72 | .74 | .16 | .62 | .28 | .74 | .32 | .50 |
| | | | $G_1^*$ | 1.0 | .75 | .99 | .93 | 1.0 | .72 | .92 | .98 | .70 | .90 | .80 | .96 | .64 | .82 |
| | | 2 | $G_1^r$ | .99 | .83 | .48 | .96 | .74 | .53 | .81 | .90 | .51 | .41 | .76 | .53 | .42 | .65 |
| | | | $G_1^*$ | 1.0 | .98 | .88 | 1.0 | .97 | .74 | .95 | .99 | .87 | .83 | .97 | .90 | .65 | .89 |
| | | 3 | $G_1^r$ | .99 | .11 | .85 | .60 | .97 | .44 | .78 | 1.0 | .84 | .76 | .98 | .93 | .54 | .89 |
| | | | $G_1^*$ | 1.0 | .36 | .93 | .91 | .99 | .53 | .85 | 1.0 | .94 | .87 | .99 | .98 | .63 | .92 |

Table C-3: Empirical power of bootstrap recursive and non-recursive $G_1$ tests, $[m = 1,\ \rho = 0,\ \lambda_1^0 = 0.5]$

| T | $\theta$ | DGP | Test | Model 1: $\delta$ | | | Model 2: $\delta$ | | Model 3: c | | Model 1: $\delta$ | | | Model 2: $\delta$ | | Model 3: c | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $\delta/c$ | 1 | 1/3 | 3 | 1/3 | 3 | 0 | 10 | 1 | 1/3 | 3 | 1/3 | 3 | 0 | 10 |
| | | | | $\alpha = 0.5\ /\alpha_1 = 0.2, \alpha_2 = 0.9$ | | | | | | | $\alpha = 0.7/\alpha_1 = 0.9, \alpha_2 = 0.2$ | | | | | | |
| 200 | 0.5 | 1 | $G_1^r$ | .92 | .32 | .84 | .63 | .91 | .45 | .75 | .76 | .28 | .53 | .44 | .68 | .33 | .56 |
| | | | $G_1^*$ | .99 | .74 | .95 | .90 | .99 | .64 | .90 | .93 | .69 | .74 | .79 | .88 | .55 | .82 |
| | | 2 | $G_1^r$ | .95 | .89 | .46 | .93 | .70 | .54 | .82 | .77 | .62 | .38 | .72 | .50 | .41 | .64 |
| | | | $G_1^*$ | .99 | .96 | .81 | .98 | .90 | .66 | .92 | .93 | .82 | .72 | .90 | .77 | .56 | .80 |
| | | 3 | $G_1^r$ | .97 | .32 | .79 | .80 | .95 | .45 | .80 | .99 | .78 | .37 | .96 | .81 | .45 | .87 |
| | | | $G_1^*$ | .99 | .51 | .81 | .91 | .97 | .49 | .87 | 1.0 | .78 | .55 | .97 | .92 | .50 | .90 |
| | 0.8 | 1 | $G_1^r$ | .88 | .58 | .75 | .75 | .85 | .54 | .76 | .82 | .55 | .56 | .69 | .77 | .45 | .71 |
| | | | $G_1^*$ | .96 | .81 | .75 | .89 | .91 | .61 | .86 | .93 | .77 | .61 | .83 | .86 | .54 | .80 |
| | | 2 | $G_1^r$ | .89 | .87 | .60 | .90 | .72 | .54 | .81 | .83 | .78 | .54 | .82 | .61 | .48 | .71 |
| | | | $G_1^*$ | .96 | .87 | .78 | .93 | .87 | .61 | .87 | .93 | .83 | .74 | .88 | .79 | .54 | .80 |
| | | 3 | $G_1^r$ | .83 | .37 | .39 | .69 | .62 | .31 | .60 | .80 | .19 | .35 | .47 | .66 | .29 | .56 |
| | | | $G_1^*$ | .82 | .41 | .37 | .70 | .59 | .31 | .60 | .80 | .17 | .38 | .46 | .68 | .31 | .56 |
| | | | | $\alpha = 0.5\ /\alpha_1 = 0.2, \alpha_2 = 0.9$ | | | | | | | $\alpha = 0.7/\alpha_1 = 0.9, \alpha_2 = 0.2$ | | | | | | |
| 400 | 0.5 | 1 | $G_1^r$ | 1.0 | .34 | 1.0 | .90 | 1.0 | .63 | .89 | .97 | .22 | .93 | .59 | .97 | .53 | .78 |
| | | | $G_1^*$ | 1.0 | .82 | 1.0 | .99 | 1.0 | .78 | .97 | 1.0 | .73 | .97 | .94 | .99 | .70 | .91 |
| | | 2 | $G_1^r$ | 1.0 | 1.0 | .64 | 1.0 | .96 | .71 | .97 | .99 | .95 | .49 | .99 | .77 | .60 | .86 |
| | | | $G_1^*$ | 1.0 | 1.0 | .93 | 1.0 | .99 | .79 | .98 | 1.0 | .97 | .85 | 1.0 | .95 | .70 | .94 |
| | | 3 | $G_1^r$ | 1.0 | .62 | .99 | .99 | 1.0 | .61 | .96 | 1.0 | 1.0 | .75 | 1.0 | .99 | .62 | .97 |
| | | | $G_1^*$ | 1.0 | .77 | .99 | 1.0 | 1.0 | .64 | .97 | 1.0 | 1.0 | .83 | 1.0 | 1.0 | .64 | .97 |
| | 0.8 | 1 | $G_1^r$ | .98 | .53 | .98 | .83 | .99 | .67 | .89 | .94 | .50 | .91 | .71 | .96 | .61 | .83 |
| | | | $G_1^*$ | .99 | .81 | .97 | .95 | 1.0 | .74 | .95 | .98 | .77 | .91 | .87 | .98 | .69 | .89 |
| | | 2 | $G_1^r$ | .99 | .98 | .66 | 1.0 | .87 | .70 | .93 | .94 | .96 | .60 | .98 | .73 | .62 | .86 |
| | | | $G_1^*$ | 1.0 | .98 | .86 | .99 | .94 | .75 | .95 | .98 | .96 | .80 | .98 | .87 | .68 | .90 |
| | | 3 | $G_1^r$ | .99 | .60 | .82 | .96 | .96 | .48 | .85 | .99 | .68 | .58 | .95 | .95 | .47 | .85 |
| | | | $G_1^*$ | .99 | .65 | .80 | .96 | .95 | .48 | .85 | .99 | .66 | .65 | .95 | .95 | .47 | .84 |

## Supplement D: Additional Empirical Results

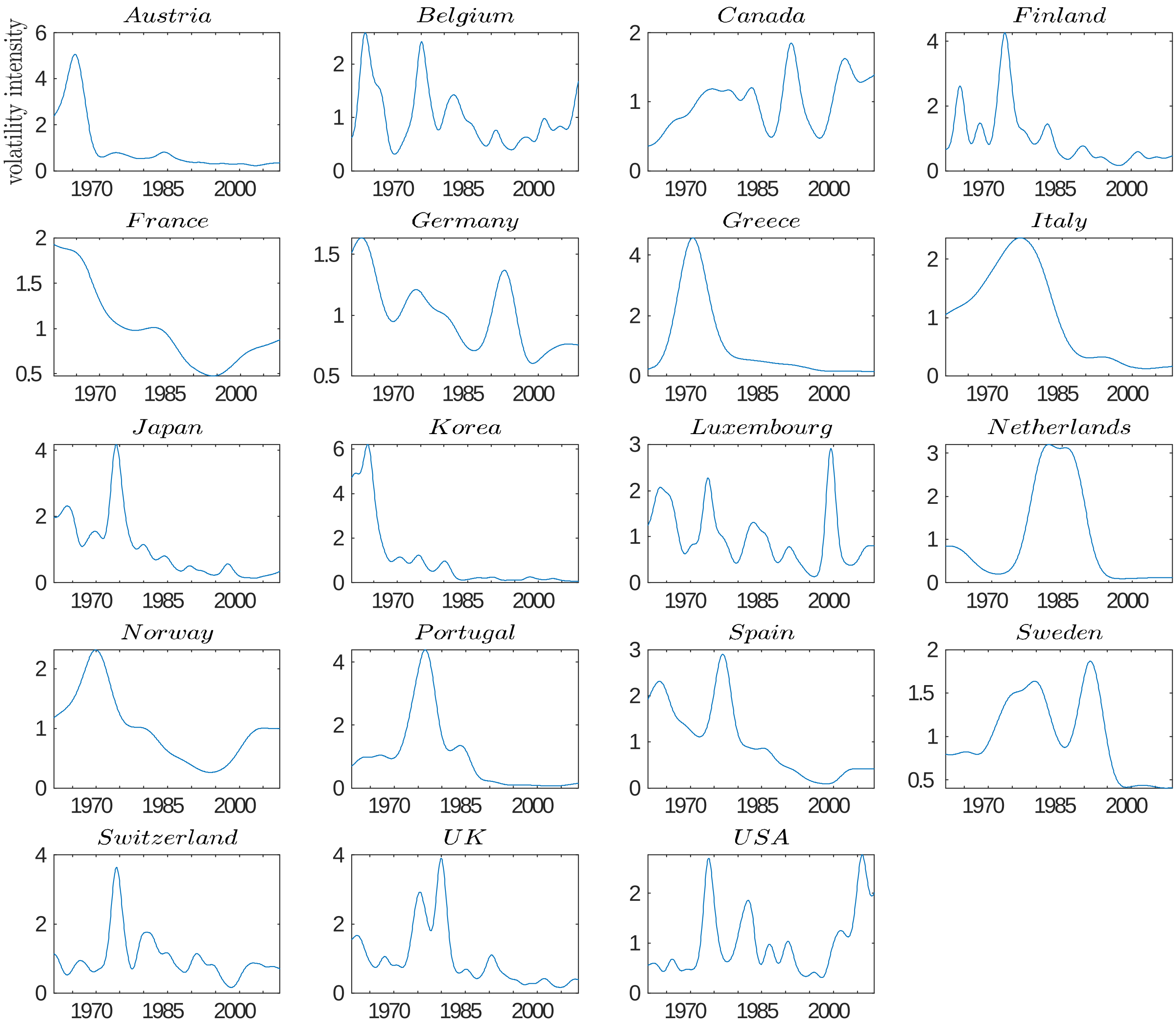


Figure D-1: Nonparametric volatility estimates (% of the total) of OECD inflation rates.

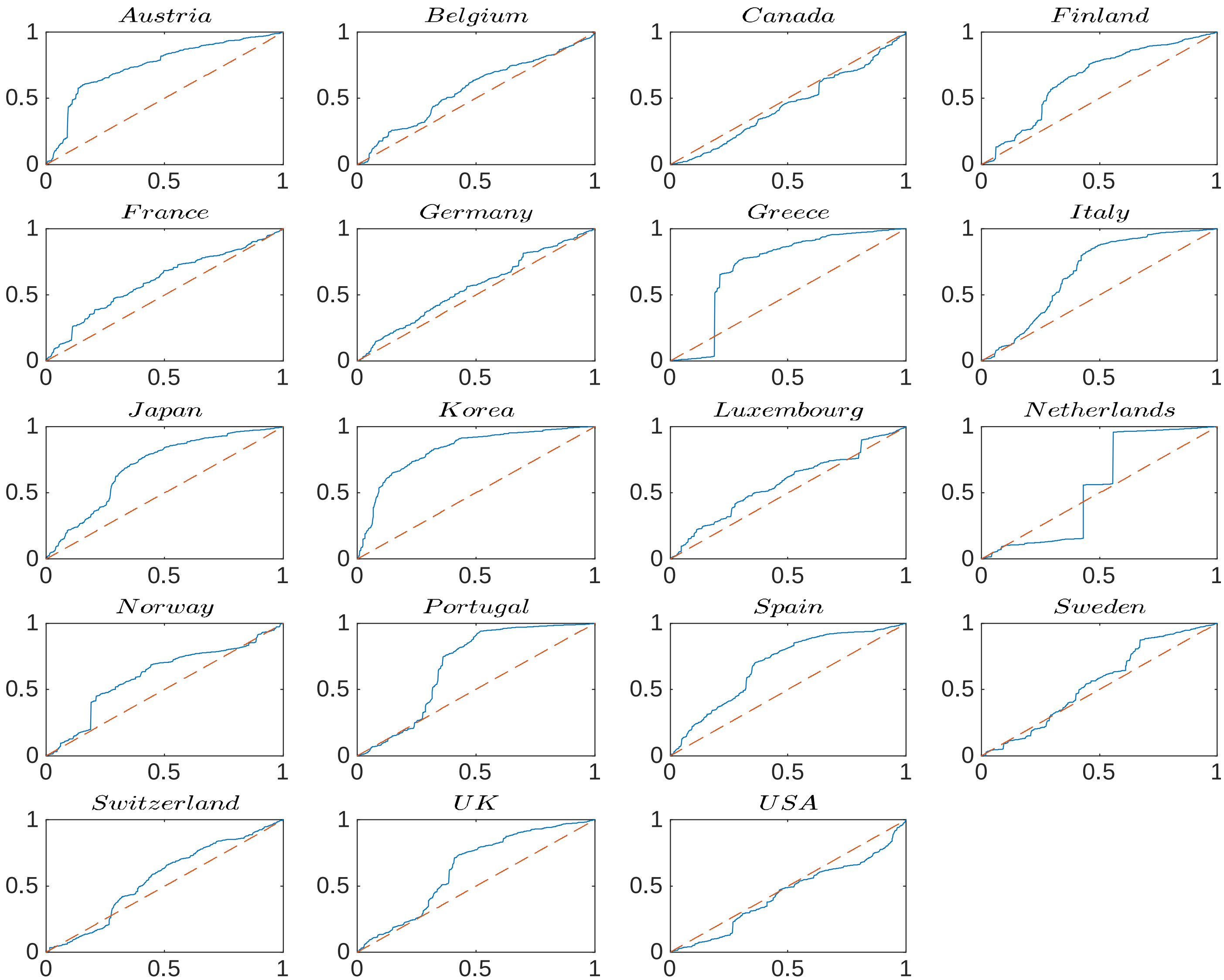


Figure D-2: Estimated variance profile of OECD inflation rates.

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